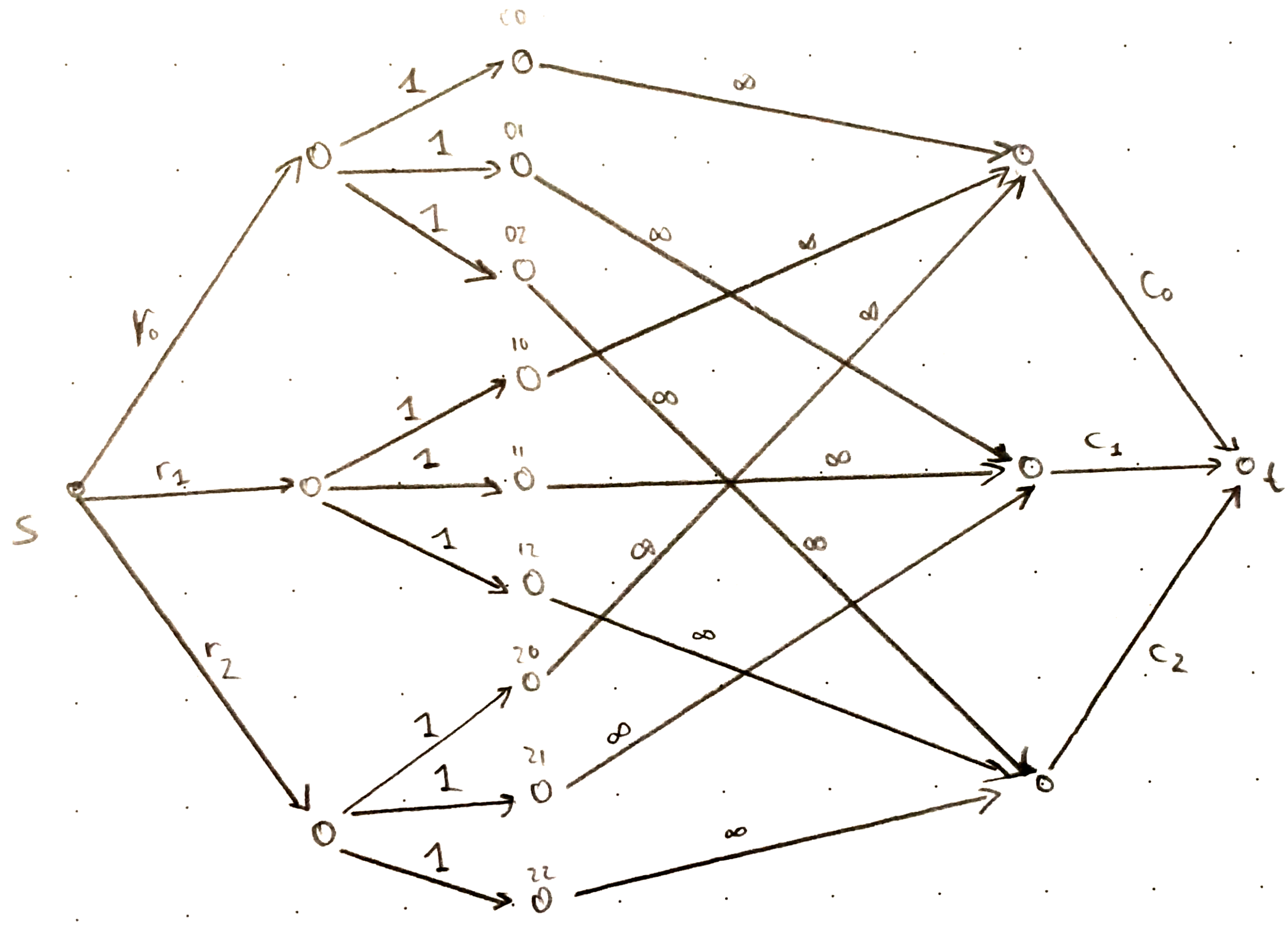
Problem 3

Subproblem a

Basic Idea



- We create a flow network for the table. The table entries can be rounded to integers by satisfying all column and row constraints iff the max flow of the graph is the sum of all column/row sums.
- We can reconstruct the rounding by evaluating the flow values.
- For every cell entry, the value can either be 0 or 1 (edges with capacity 1). A flow value of 1 means that the cell entry is 1.
- Edges with flow r_i and c_i ensure that the column/row sums are not exceeded. If the max flow value is smaller than the sum of all column/row sums, then at least one row/column sum is too small.

Graph Construction

- Create a source s and a sink t.
- Create a vertex v_{r_i} for every row i and a vertex v_{c_i} for every column i.
- Create edges (s, v_{r_i}) with capacities r_i where r_i is the row sum of row i.
- Create edges (v_{c_i}, t) with capacities c_i where c_i is the column sum of column i.
- Create a vertex $c_{a,b}$ for every cell where a is the cell's row and b is the cell's column.
- Create edges $(v_{r_i}, c_{i,b})$ for every b and every row i with capacity 1.
- Create edges $(c_{a,i}, v_{c_i})$ for every a and every column i with capacity ∞.

Algorithm

- Build the flow graph G for the input table.
- Run Ford-Fulkerson on G to determine the max flow.
- Extract the solution from the max flow[1].
  - Round $c_{a,b}$ up (use 1) if $f((v_{r_a}, c_{a,b})) = 1$.
  - Round $c_{a,b}$ down (use 0) if $f((v_{r_a}, c_{a,b})) = 0$.
- Output the rounded values $c_{a,b}$.

Proof

- **Integrality constraints:** since all capacity values are integers, Ford-Fulkerson assigns only integer flow values. Therefore, for every combination of a and b, $c_{a,b}$ is always an integer. Having calculated the max flow, we can always extract the information whether to round a number up or down.
- **Rounding constraints:** since all edges $(v_{r_i}, c_{i,b})$ have a capacity of 1 and integrality is ensured, we can conclude that the max flow assigns only values 0 or 1 to each cell. Therefore, we can be sure that this algorithm does only round numbers up or down but not assign other numbers to cells[2].
- **Row constraints:** for every row i, all $c_{i,b}$ (for all b) are connected to the source via edges (s, v_{r_i}), with a capacity of the respective row sum. Therefore, we can be sure that the sum of all rounded cells in row i cannot exceed r_i. We also know that there is no possible rounding, if the max flow is smaller than $\sum_i r_i$. In that case, at least one of the edges (s, v_{r_i}) has a remaining capacity that was not used by the max flow. Therefore, the row sum is too small.
- **Column constraints:** we can make the same argument for column sums, by examining the edges (v_{c_i}, t).
- **Summary:** the algorithm finds a valid rounding if there exists one, because all cell numbers are either rounded to 0 or 1, row and column constraints are ensured and we can extract the rounding from the max flow/residual graph after running the Ford-Fulkerson algorithm.

[1] $f(e)$ is the flow value for e in the maximum flow.

[2] Assumption: the problem description says that all cell values are between 0 and 1. We assume, that the bounds are excluding, i.e. there is no cell with a value of 0 or 1. If this is allowed, we can adapt the algorithm to cope with this situation (see later subsection).

Runtime

- Building the graph can be done in $\mathcal{O}(n)$, where n is the number of cells in the table, because we have to add n vertices in the middle, $2n$ edges connecting there vertices, and $2\sqrt{n}$ vertices for row/column sums[3] together with the same number of edges connecting these vertices with the source and the sink.
- The Edmont-Karp algorithm, a variation of the ford fulkerson algorithm, can find the max flow in $\mathcal{O}(VE^2)$, where V is the number of vertices and E is the number of edges in the graph. In our case, $V = \mathcal{O}(n)$ and $E = \mathcal{O}(n)$ (see previous argument), therefore the runtime is $\mathcal{O}(n^3)$.
- For extracting the solution, we need to examine n edges.
- The overall runtime of the algorithm is $\mathcal{O}(n^3)$.

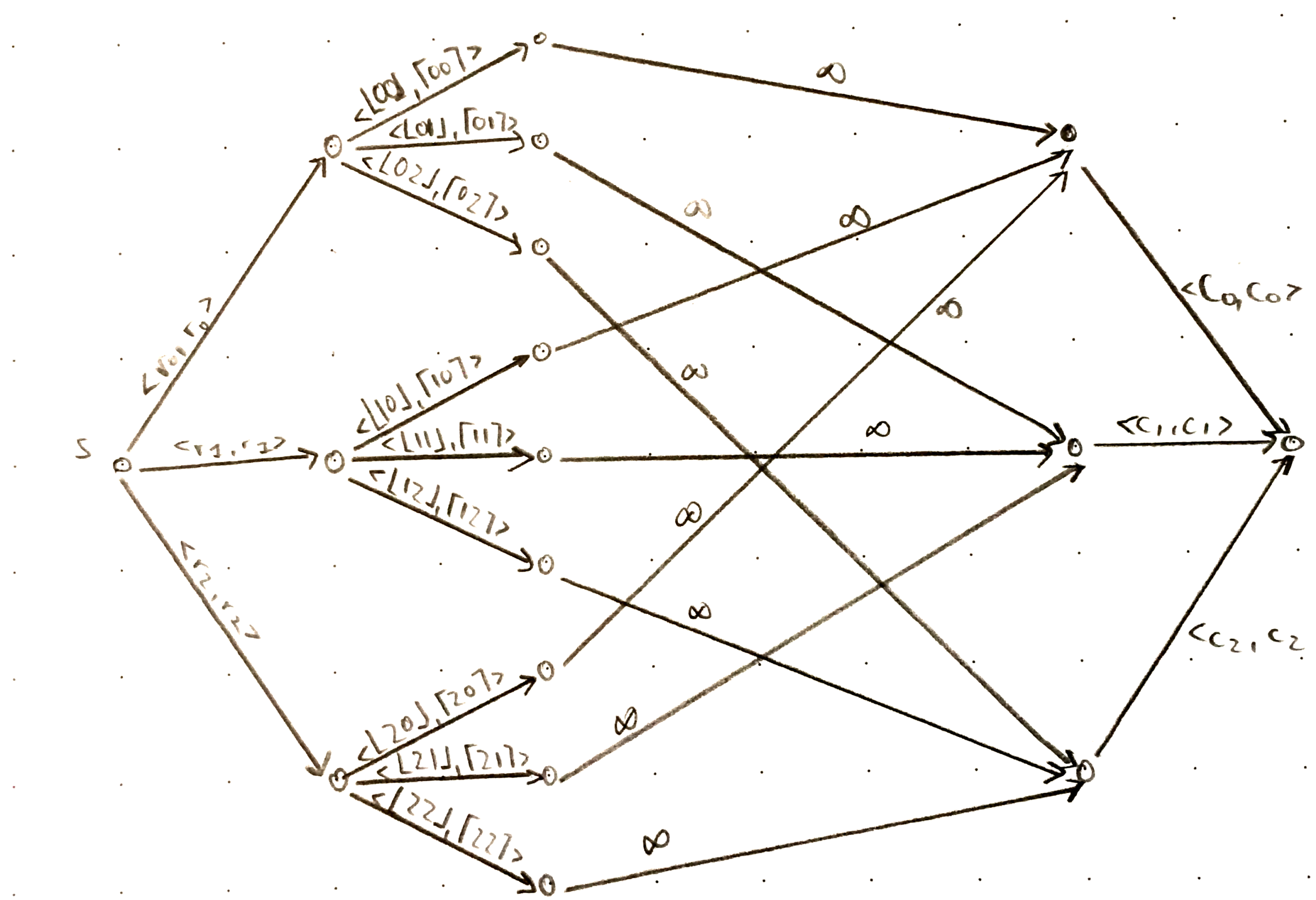
Variation: Inclusive lower/upper bound

The algorithm can be modified to support values of 0 and 1 inside the table. Instead of showing the exact changes necessary here, take a look at subproblem b, which is a more general problem and supports inclusive lower/upper bounds, i.e. integers as cell values.

[3] We assume that the number of rows and columns is equal. Even if this is not the case, this number can never exceed n.

Subproblem b

Basic Idea



- We use a modified version of max flow that supports lower bounds on the flow values, in addition to capacity constraints. We will show how this can be reduced to the original max flow problem.
- The graph is similar to subproblem a, but instead of putting a capacity constraint of 1 on all edges $(v_{r_i}, c_{i,b})$, we use a lower bound of $\lfloor o_{i,b} \rfloor$ and an upper bound (capacity constraint) of $\lceil o_{i,b} \rceil$, where $o_{i,b}$ is the original value of the cell (i, b) in the table.
- We force the connecting edges of the source and the sink to be the row/column sums by putting the same lower and upper bound of r_i or c_i respectively on the edge flow.
- The problem is solvable, if there is a flow that satisfies all constraints. Note, that the notion of a max flow does not make sense here, because all of s's outgoing edges are forced to be a specific value.

Graph Construction

- Build the graph exactly as in subproblem a but with the following modifications.
- For every edge $(v_{r_i}, c_{i,b})$, use $\lfloor o_{i,b} \rfloor$[4] as a lower bound and $\lceil o_{i,b} \rceil$ as an upper bound.

[4] In the graph illustration, we just wrote 00 instead if $o_{0,0}$.

- For every edge (s, v_{r_i}), use r_i as a lower and as an upper bound.
- For every edge (v_{c_i}, t), use c_i as a lower and as an upper bound.

Algorithm

We assume at this point that we have an algorithm that can solve the flow problem[5], i.e. assign flow values to all edges, such that all constraints are satisfied. We are not looking for a max flow here, but only some assignment, that satisfies flow conservation, capacity constraints and minimum flow constraints.

- Build the flow graph G for the input table.
- Assign flow values to the graph, such that all constraints are satisfied, using the algorithm described in the next subsections.
- Extract the solution from the flow values.
  - Round $c_{a,b}$ up if $f((v_{r_a}, c_{a,b})) = \lceil o_{a,b} \rceil$.
  - Round $c_{a,b}$ down if $f((v_{r_a}, c_{a,b})) = \lfloor o_{a,b} \rfloor$.
- Output the rounded values $c_{a,b}$.

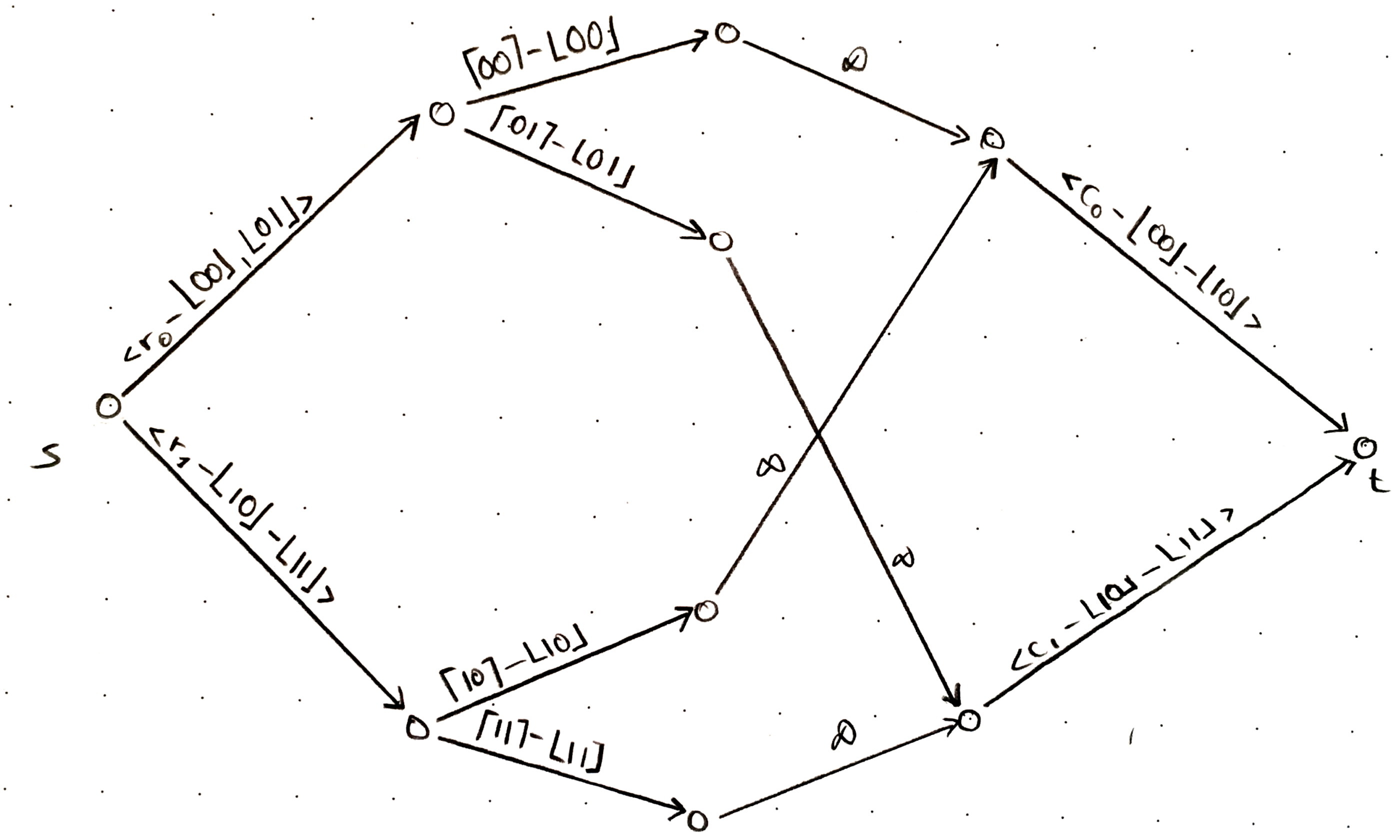
Proof

- The proof is similar to subproblem a's proof.
- **Integrality constraints:** see subproblem a.
- **Rounding constraints:** similarly to subproblem a, we assign only the values $\lfloor o_{a,b} \rfloor$ or $\lceil o_{a,b} \rceil$, because these are the lower/upper bounds, their difference is at most 1 and we already showed integrality. Note, that this version of the algorithm allows values in the table to be integers (in contrast to the algorithm shown in subproblem a). Therefore, we can use this algorithm to solve subproblem a where the lower/upper bounds are inclusive, i.e. 0 and 1 are allowed as table cell values.
- **Row constraints:** we enforce that the sum of the cells per row is the row sum for that row. Therefore, we do not have to calculate the flow value of the graph. For a single row it is sufficient to check, whether the upper/lower bounds are satisfied. These constraints must be satisfied, because this is ensured by the algorithm that we ran in the second step.
- **Column constraints:** the same argument can be made for column constraints.

Reduction to Max-Flow

The following graph shows how to reduce the previous graph with lower bounds to the classical max flow problem (for a smaller example).

[5] How this actually works is shown in one of the next subsections.



- For every edge $(v_{r_a}, c_{a,b})$ with a lower bound of l and an upper bound of r, remove l from every edge on every s-t path that uses this edge. Note, that there is only one that path. The lower bound is now 0, thus we removed a lower bound.
- The resulting graph still has edges (s, v_{r_a}) and (v_{c_b}, t) that enforce a specific value (by using the same upper and lower bound). We remove the lower bound.
- If and only if the resulting graph has a max flow of $\sum_i r_i - \sum_{a,b} \lfloor o_{a,b} \rfloor$, then the problem is solvable. By subtracting the lower bound from every path, we could still satisfy the constraints in the original graph by just adding the lower bound to every edge on this path. When we achieve the stated max flow value, then all of s's outgoing edges and all of t's incoming edges have a residual (forward) capacity of 0, i.e. there is no capacity left. Therefore, after adding the subtracted values again, we would flow the row/column sums using these edges. This is necessary for a valid solution.
- This is how we construct the original flow values for edges $(v_{r_a}, c_{a,b})$: if flow in the new graph is 0, then we flow the lower bound. If the flow is 1, then we flow the upper bound. Note, that the difference between upper and lower bound is at most 1.

Runtime

The runtime of this problem is the runtime of problem a plus the runtime required for transforming the graph. Changing the edge capacities/lower bounds can be done in $\mathcal{O}(E)$, where E is the number of edges. From the argument in subproblem a we know that there are $\mathcal{O}(n)$ edges. Therefore, the overall runtime is still $\mathcal{O}(n^3)$.

Subproblem c

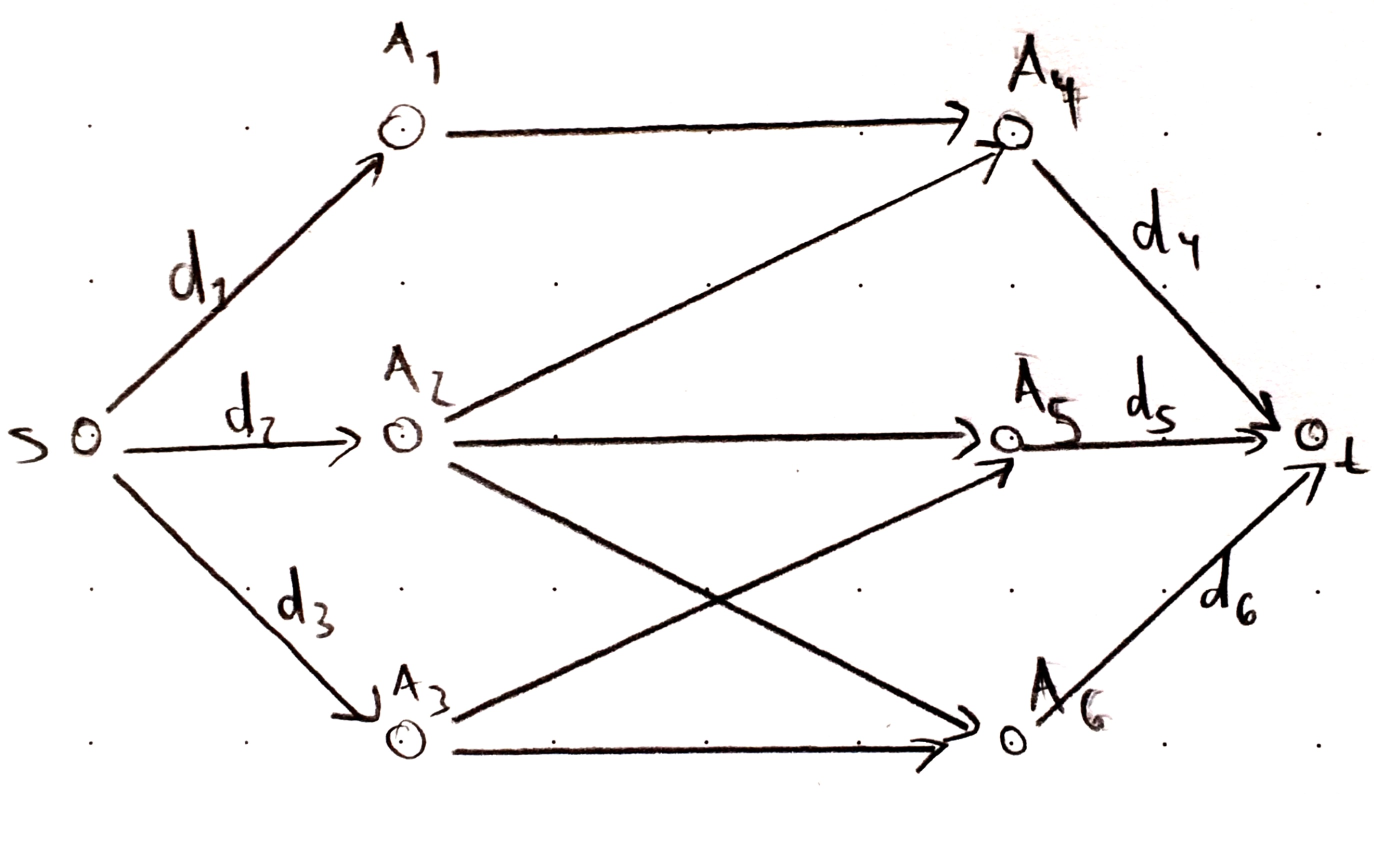
Note, that subproblem a is just a special case of subproblem b. Therefore, if we prove the assumption for subproblem b, we have automatically proven it for subproblem a.

We prove that the max flow/min cut in the modified graph without lower bounds is always $\sum_i r_i - \sum_{a,b} \lfloor o_{a,b} \rfloor$. In that case, there is always a valid solution for the graphs with lower bounds (see argumentation in subproblem b). Therefore, we can always round the numbers, such that the constraints are satisfied.

- We claim that $C = (A, B)$, $A = \{s\}$, $B = V - A$ is a min cut.
- If we add one of the vertices v_{r_a}, then the cut capacity can only become bigger (or stay the same). We can prove this by examining the change of the cut capacity: $(\sum_i \lceil o_{a,i} \rceil - \lfloor o_{a,i} \rfloor) - (r_a - \sum_i \lfloor o_{a,i} \rfloor) = \sum_i (\lceil o_{a,i} \rceil) - r_a$. We know that this term must be positive, because by definition $r_a = \sum_i o_{a,i}$ and the ceiled value is always bigger than or equal to the value.
- If we add another vertex $c_{a,b}$ in addition, we also have to add the vertex v_{c_b}, because the connecting edge has infinite capacity.
- If we additionally add another vertex v_{c_b}, we can not compensate this. We can prove this by examining the total change of adding both vertices: $c_b - \sum_i \lfloor o_{i,b} \rfloor + \sum_{i \neq b} (\lceil o_{a,i} \rceil - \lfloor o_{a,i} \rfloor) - (r_a + \sum_i \lfloor o_{a,i} \rfloor) \geq 0$ (expand r_a and c_b similarly to previous part).
- Therefore, we have no way of achieving a smaller minimum cut. Note, that the cut $C = (A, B)$ with $B = \{t\}$, $A = V - B$ has the same minimum cut value.

Problem 5

Basic Idea



- We generate a flow graph as illustrated in the example above.
- The middle part of the graph (everything without s and t) is the original graph, but with directed edges. The capacities for these edges are 1[6].
- We connect the left vertex set with the source and give each each edge (s, A_i) a capacity of d_i, where d_i is the degree of A_i in the spanning subgraph. We do the same for the right vertex set and the sink.
- There exists such a spanning subgraph, if and only if the max flow of this graph is the sum of all d_j, where j is the set of vertices on the left side.
- We can extract the spanning subgraph by taking only the edges with a flow of 1.
- We can partition the bipartite graph into two vertex sets with DFS. Note, that a graph is bipartite if and only if its chromatic number is 2. Therefore, the DFS can just assign the opposite color when exploring a new vertex.

Graph Construction

- Add a source s and a sink t.
- Let $V = (V_1, V_2)$ be the original vertex set.
- Add all vertices V.
- Add an edge (a, b) for all $a \in V_1$, $b \in V_2$, $\{a, b\} \in E$ with capacity 1.
- For all $a \in V_1$, add an edge (s, a) with capacity d_a. For all $b \in V_2$, add an edge (b, t) with capacity d_b.

Necessary Conditions

- $G = (V, E)$ is bipartite, therefore we can write $V = (V_1, V_2)$. $|V_1| = |V_2|$ is a necessary condition. Otherwise, we would require V_1 to have more outgoing edges than V_2 has incoming edges or V_2 to have more incoming edges than V_1 has outgoing edges.

Algorithm

- Split V in sets V_1 and V_2 with $V_1 \cap V_2 = \emptyset$ and $\forall a \in V_1, b \in V_2 : \{a, b\} \notin E$ using DFS: when exploring a new node, assign the node the opposite color of the current node (there are only two colors). The initial node can be an arbitrary color. Split V into V_1 and V_2 according to their colors.
- Build the flow network.
- Calculate the max flow with Ford-Fulkerson algorithm.
- If the max flow is smaller than $\sum_{i \in V_1} d_i$, then output *no solution*.
- Else, output all edges (a, b) with $a \neq s$ and $b \neq t$ and $f((a, b)) = 1$.

[6]note, that we can arrange the vertices in such a way because the graph is bipartite.

Proof

- **Integrality constraints:** Ford-Fulkerson generates a max flow with integer flow values, because all capacity values are integers. Therefore, the flow values for edges (a, b) with $a \in V_1$ and $b \in V_2$ are 0 or 1 and we can extract the solution from that.

- **Degree constraints:** For every vertex $a \in V_1$ in the left vertex set, we cannot use more than d_a outgoing edges, because the capacity of the connecting edge (s, a) is d_a. Similarly, for every vertex $b \in V_2$, we cannot use more than d_b incoming edges. When the max flow is $\sum_{a \in V_1} d_a = \sum_{b \in V_2} d_b$, we can be sure that we used d_i edges for every vertex V_i (because of flow conversation).

- We can always partition V into V_1 and V_2, because a graph is bipartite if and only if it is two-colorable.

Runtime

- Generating sets V_1 and V_2 with DFS: $\mathcal{O}(|V| + |E|)$.

- Build the flow network. This involves adding $|V|$ connecting edges and setting capacity constraints for $|E|$ edges. Therefore, the runtime is bounded by $\mathcal{O}(|V| + |E|)$

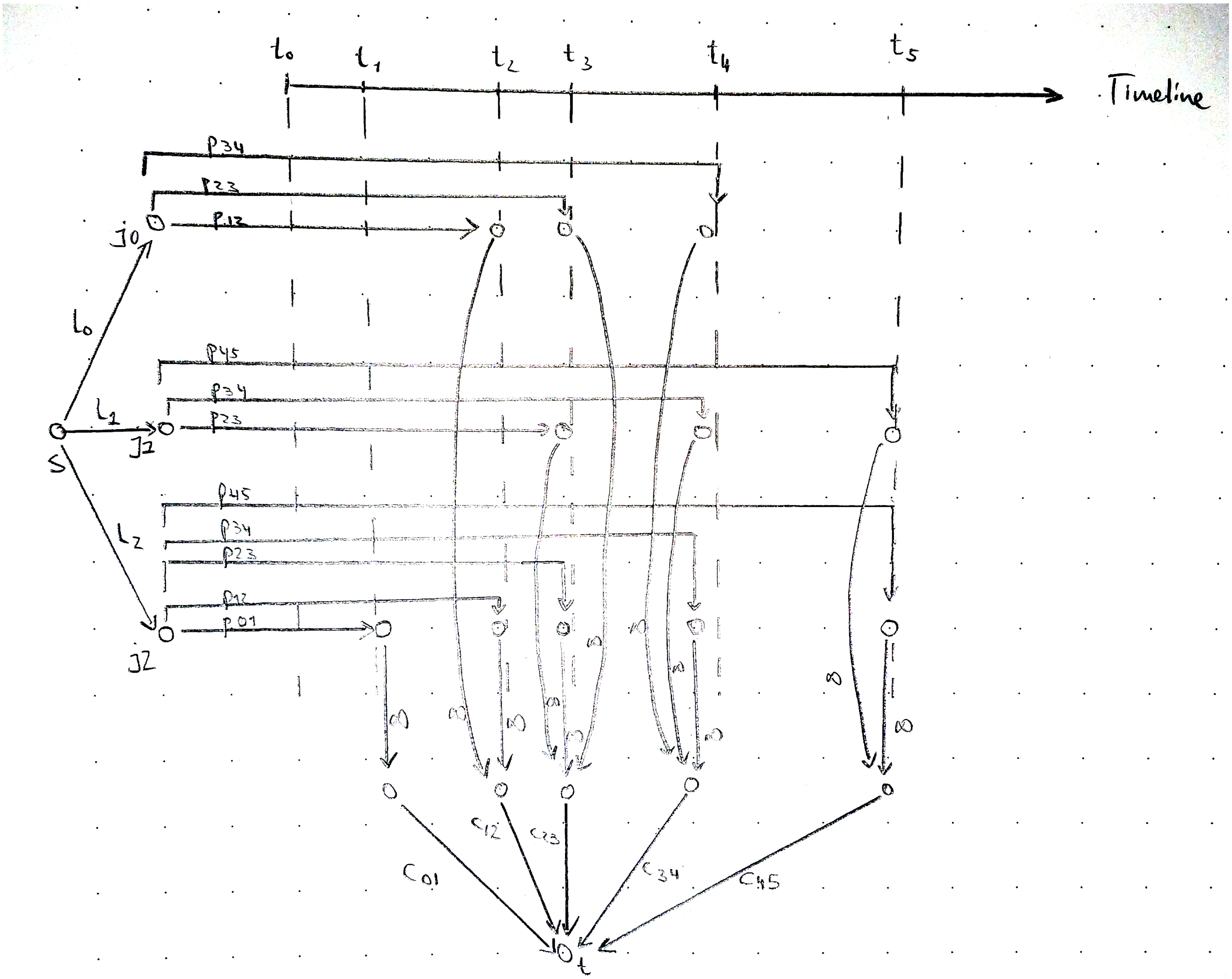
- Generating the max flow with the Edmonds-Karp algorithm in $\mathcal{O}(|V||E|^2)$

- Extracting the solution by examining $\mathcal{O}(|E|)$ edges.

- The overall runtime is $\mathcal{O}(\max\{|V| + |E|, |V||E|^2\})$.

Problem 1

Basic Idea



- We precalucate a set of time points t_i where *something* changes, i.e. where a processor becomes available, a processor vanishes or a job becomes available.
- At every t_i, we redistribute a subset of all available jobs onto the available processors.
- For every time interval t_{i-1} to t_i, we precalculate the processor capacity $c_{i-1,i}$, i.e. the processing power of all available processors. For instance, if two processors are available in the interval t_{i-1} to t_i with length $[4.5; 6)$, then the processor capacity for this interval is $c_{i-1,i} = 3$.
- We connect every job with the source. The capacity of the edge is the duration of the job.
- We connect every job with every possible end point of a time interval (by creating new vertices k for every job). For instance, if job 1 is available from t_2 to t_5, we connect it with t_3, t_4 and t_5, because it could run in any one of the intervals 2-3, 3-4, 4-5. Each edge's capacity is the length of the interval

($p_{i-1,i}$). This ensures that we do no use more processing power than one processor can provide in this interval.

- For every time point t_i, we connect the vertices of all jobs at this time point with an intermediate vertex. This vertex is connected to the sink with a capacity value of $c_{i-1,i}$. This ensures that we do not schedule too many processes.
- The problem has a solution if and only if the max flow is $\sum_i l_i$, where l_i is the length of job i. By examining the flow values of the edges with capacities of $p_{a,b}$ we can determine in which intervals a job is supposed to run. To generate a concrete schedule, we have to do a post-processing step.

Graph Construction

- Create a source s and a sink t.
- Create a vertex j_i for every job i and connect it to the source with a capacity of l_i (direction: towards j_i).
- For every job i, create a set of vertices $k_{i,j}$, where i is available in an interval t_{j-1} to t_j. I.e. the smallest t_{j-1} represents the time point where i becomes available and the largest t_j represents the deadline for the job.
- Connect every j_i with all $k_{i,m}$ with a capacity that is the duration of the interval $m-1$ to m (in other words: the processing power of one processor in that time interval).
- Create vertices w_i for every time point i. These vertices sum up all the work that is done in one interval.
- Create edges $(k_{i,j}, w_j)$ for all jobs i, with capacity of ∞.
- Create edges (w_i, t) with a capacity of $c_{i-1,i}$.

Algorithm

- Calculate all time points t_i by creating list of all times when a processor becomes available or vanishes and when a job becomes available. Sort that list.
- For every time interval t_{i-1} to t_i calculate the processor capacity by checking which processors are available in every interval and multiplying this number with the length of the interval.
- Generate the flow graph G. If the max flow is smaller than $\sum_i l_i$, return *no solution*.
- Calculate the max flow of G with the Ford-Fulkerson algorithm.
- For every interval t_{i-1} to t_i and every job m, generate a mapping of running times. The runtime of job m in that interval is the flow value of the edge between the vertices j_m and $k_{m,i}$.
- For every interval, generate a concrete schedule: use the algorithm described below.

$cpu \leftarrow 1$
$time \leftarrow 0$
for all $(job, runtime) \in mapping$ **do**
  if $time + runtime \leq intervalLength$ **then**

$schedule(job, cpu, time, time + runtime)$
$time \leftarrow time + runtime$
else
$schedule(job, cpu, cpu, intervalLength)$
$time \leftarrow runtime - (intervalLength - time)$
$cpu \leftarrow cpu + 1$
$schedule(job, cpu, 0, time)$
end if
end for

The function *schedule* schedules a job on a specific processor for a given time span (start time and end time). Note, that the algorithm might schedule a job on two different processors, but a single job is never run in parallel on multiple processors, because the length of a job run (*runtime*) is never bigger than the length of the interval[7].

Proof

- The number of processors stays constant in every interval and no new jobs are added. Therefore, we can assume that we have a single processor with a bigger amount of computing power, as long as no job gets scheduled for more than the interval length (parallel computation is not allowed). We show later, that we can then always generate a concrete valid scheduling.
- Every job is scheduled: if the max flow is $\sum_i l_i$, then every job i runs exactly for a time of l_i during a number of intervals, because due to flow conservation the computing time must be delegated to these intervals.
- Every interval t_{i-1} to t_i cannot get more than $c_{i-1,i}$ computing load (capacity constraints on edges that lead to t). In addition, no job in that interval is scheduled longer than the interval length. Therefore, we can always generate a concrete schedule for every interval by scheduling the jobs after each other and choosing a new processor once the current processor has no computing time left in the current interval. The rest of the current job might be scheduled on the new processor, resulting in two different run time spans for a single job. However, these two spans can never overlap, because the total computing time for a single job does not exceed the interval length because of the capacity constraints $p_{i-1,i}$.

Runtime

Let r be the number of processors and e be the number of jobs.

- Calculating points t_i: the list contains $2r+e$ time points. Therefore, sorting takes $\mathcal{O}((2r+e)\log(2r+e))$ time.
- Calculating the processor capacity for every time interval: there are $2r+e-1$ time intervals and for every interval, we need to check r processors. Therefore, this takes $\mathcal{O}(r^2+er)$ time.
- Generate the graph G: we create vertices for every job, every time interval and a vertex for some time intervals in every job. Therefore, we create $\mathcal{O}(er)$ vertices and no more than $\mathcal{O}(e^2r^2)$ edges (assuming that we generate a node between every pair of vertices).

[7] See proof for more details.

- Calculate the max flow with the Edmonds-Karp algorithm: since we have $\mathcal{O}(er)$ vertices and $\mathcal{O}(e^2r^2)$ edges, the Edmonds-Karp algorithm runs in $\mathcal{O}(e^5r^5)$.
- Generating the concrete schedule from the mapping: in the worst case, every job is scheduled very shortly in every interval, therefore we have to examine $\mathcal{O}((2r+e) \cdot e) = \mathcal{O}(re + e^2)$ job time spans.
- The overall runtime complexity of the algorithm is therefore $\mathcal{O}(e^5r^5)$ ($e \geq 1$ and $r \geq 1$)[8]. Therefore, the algorithm is polynomial.

Problem 2

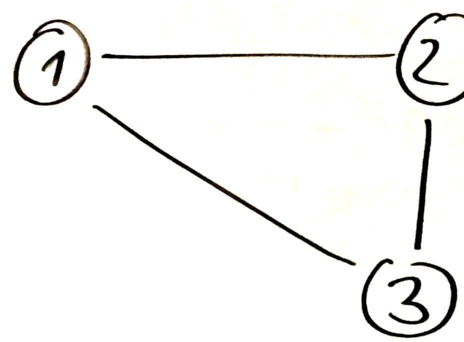
Relation to Baseball Elimination

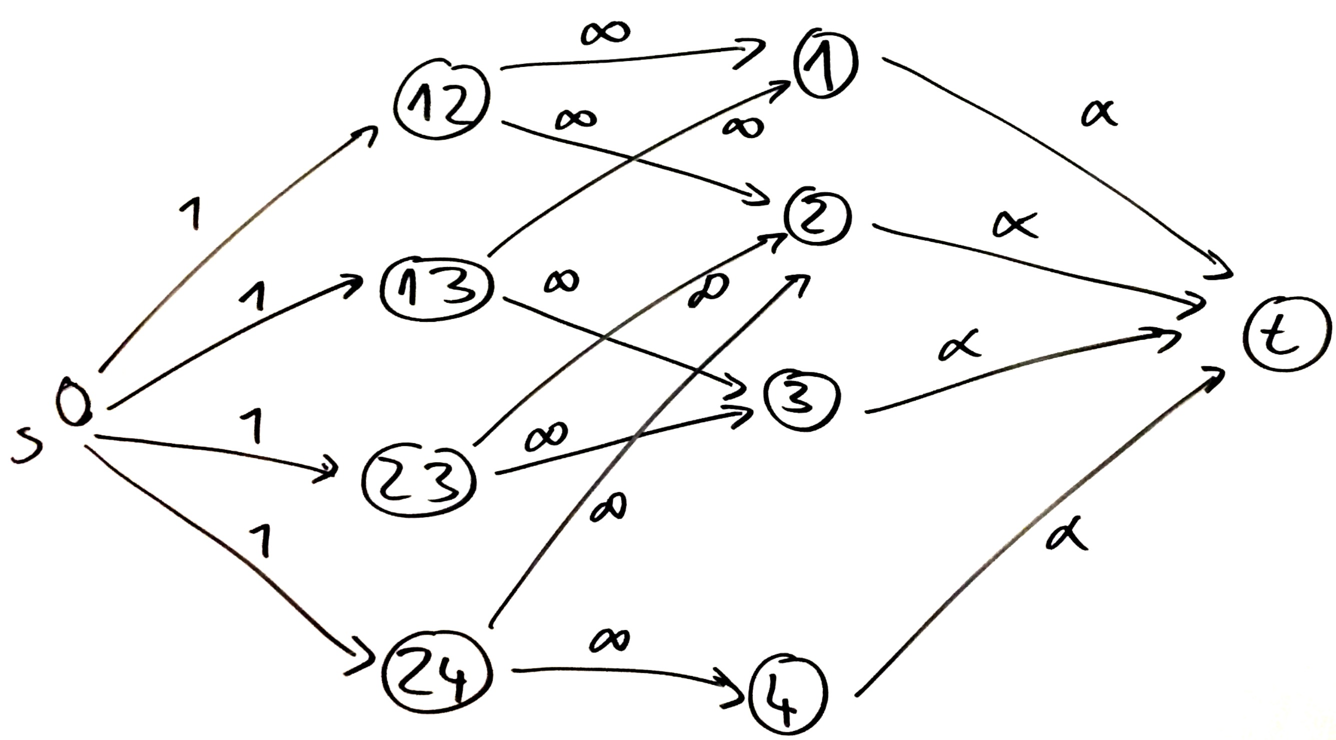
- This problem is related to the Baseball elimination problem and we use a similar flow graph to solve it.
- Every vertex in the graph corresponds to a baseball team and every edge corresponds to a game to be played between two teams.
- No team has won a game so far and every team plays against another team only once.
- A subset of teams S (vertices) can eliminate another team t, if its number of remaining games against each other divided by $|S|$ is greater than the games to be played by t (we call that number r_t), because in that case at least one team in S must have more than r_t wins.
- Apply this idea to the graph cohesiveness problem: a set of vertices S can eliminate another vertex t, if $\frac{e(S)}{|S|} > d_t$, where d_t is the degree of vertex t.

[8] There is most likely a much better upper bound, but the problem description just asked for a polynomial algorithm.

Subproblem a

Basic Idea





- We build a flow graph as shown in the figure above. There is a subset S with a cohesiveness bigger than α iff the max flow value is smaller than $|E|$
- Let the vertices ab on the left side represent matches and the vertices a on the right side represent teams. Let every team have zero wins and every match ab happen exactly once if the vertex is present. If the max flow value is smaller than $|E|$, this means that not all wins of the $|E|$ matches can be distributed in such a way that no team that has (only) α matches left gets eliminated.
- If a team with α matches left gets eliminated, there must be a subset S with $\frac{e(S)}{|S|} > \alpha$ (same argument as lemma 7.59 in the text book).

- The fact whether such a subset S exists, changes only for discrete values of α. There is a subset S with $\frac{e(S)}{|S|} \geq \alpha$ iff the max flow is smaller than $|E|$ for the next smaller value of α.

Graph Construction

- Add a source s and a sink t.
- For every vertex v_i in the original graph, add a vertex v_i.
- For every edge (v_i, v_j) in the original graph, add a vertex v_{ij}.
- For every vertex v_{ij} in the new graph, add an edge (s, v_{ij}) with capacity 1.
- For every vertex v_{ij} in the new graph, add two edges (v_{ij}, v_i) and (v_{ij}, v_j) with capacity ∞.
- For every vertex v_i in the new graph, add an edge (v_i, t) with capacity α.

Algorithm

Note, that the fact whether a subset S with $\alpha \geq \frac{e(S)}{|S|}$ exists, changes only at discrete values of α. The upper part of the fraction must be an integer value in $[0; |E|]$ (we cannot have more than $|E|$ edges). The lower part of the fraction must be an integer value in $[0; |V|]$ (we cannot have more than $|V|$ vertices). We can show that the difference between two different values α_1, α_2 must be at least $\frac{1}{n^2}$[9].

$$\alpha_1 - \alpha_2 = \frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 b_2 - a_2 b_1}{b_1 b_2}$$

We know that the upper part of the fraction must be an integer i with $|i| \geq 1$ and the lower part of the fraction can never be greater than n^2, because both b_1 and b_2 are bounded by n. Therefore, $|\alpha_1 - \alpha_2|$ must be at least $\frac{1}{n^2}$.

- Calculate the next smaller value $\alpha' = \alpha - \frac{1}{n^2}$
- Build the flow graph G' based on the graph G as described above with α'.
- Calculate the maximum flow value with the Ford-Fulkerson algorithm.
- If the maximum flow value is smaller than $|E|$: output *true*, otherwise output *false*.

Proof

Let us assume that the max flow value for the graph is greater or equal to $|E|$. In that case, all matches, represented by edges $(s, v_{i,j})$, can distributed to teams (represented by v_i) in such a way that no team has more than α wins. Therefore, no subset of teams can eliminate a team that has α games left. Therefore, no subset of teams of size $|S|$ can have more than $\alpha|S| = e(S)$ matches (represented by that number of edges) left. This means, that there cannot be a subset S with $\alpha < \frac{e(S)}{|S|}$.

Let us assume that the max flow for the graph is lower than $|E|$. In that case, not all matches can be distributed to teams in such a way that no team has more than α wins. Therefore, at least one team must have more than α wins. According to lemma 7.59 (that we also proved in the lecture), we can find a *certificate* (a subset S) such that $\frac{e(S)}{|S|} > \alpha$. See subproblem b for a description how to find this set.

[9] $n = |V|$.

Since the problem description is slightly modified in a way that the cohesiveness must be at least α (instead of strictly larger), we can find the answer to that question by examining the next smaller value of α. The proof for finding this next smaller value is described in the algorithm part.

Runtime

- Building the flow graph involves adding $\mathcal{O}(|V| + |E|)$ vertices and $\mathcal{O}(|V| + |E|)$ edges, since we add edges from the source, to the sink and in middle part of the graph from every v_{ij} to both v_i and v_j. We create exactly two edges per edge in G for the middle part of G'.

- Calculating the maximum flow with the Edmonds-Karp algorithm takes at $\mathcal{O}((|V| + |E|)^3)$. This is therefore also the overall runtime of the algorithm.

Subproblem b

Basic Idea

- We run the algorithm described in subproblem a for every value of α, starting with the smallest possible value. We have already shown that α can take only discrete values.

- When the algorithm tells us that, for a value of α, there is no subset S with cohesiveness greater or equal to α, we know that the previous value of α was the maximum value.

- We can extract the subset S from the minimal cut: all vertices v_i that are on the same side as the source (same technique as in baseball elimination).

- We can optimize this with binary search, resulting in fewer runs of the algorithm.

Algorithm

- Let $\alpha = 0$.

- Repeat until algorithm in subproblem a says *false*: $\alpha = \alpha + \frac{1}{n^2}$.

- Let α^* be the previous value of α (where the algorithm still sayed *true*), G^* be the graph for α^*, and G^*_r be the residual graph that was generated during the Ford-Fulkerson algorithm.

- Generate a min-cut for G^* using G^*_r. This can be done by doing a DFS starting at s. All reachable vertices are in the set A and $B = V - A$.

- Output all vertices v_i (that are on the right side in the figure illustrating G) in set A.

The algorithm can be optimized with binary search: instead of increasing α by 1, increase/decrease α by half of the binary search interval. We know that $\alpha < \frac{|E|}{1}$, in case $|S| = 1$. We can never reach this value, because there is no way to have edges with only one vertex. But we can be sure that the algorithm of subproblem a will eventually say *false* in the loop.

Proof

We already proved the algorithm of subproblem a. We still need to prove that we can extract S from the minimum cut. The proof is the same as for the baseball problem. The main idea for the proof is taken from the text book. For the proof, we take a look at the flow for the graph with the last value of α, for which the algorithm of subproblem a said *true*, i.e. the last α for which no team was eliminated.

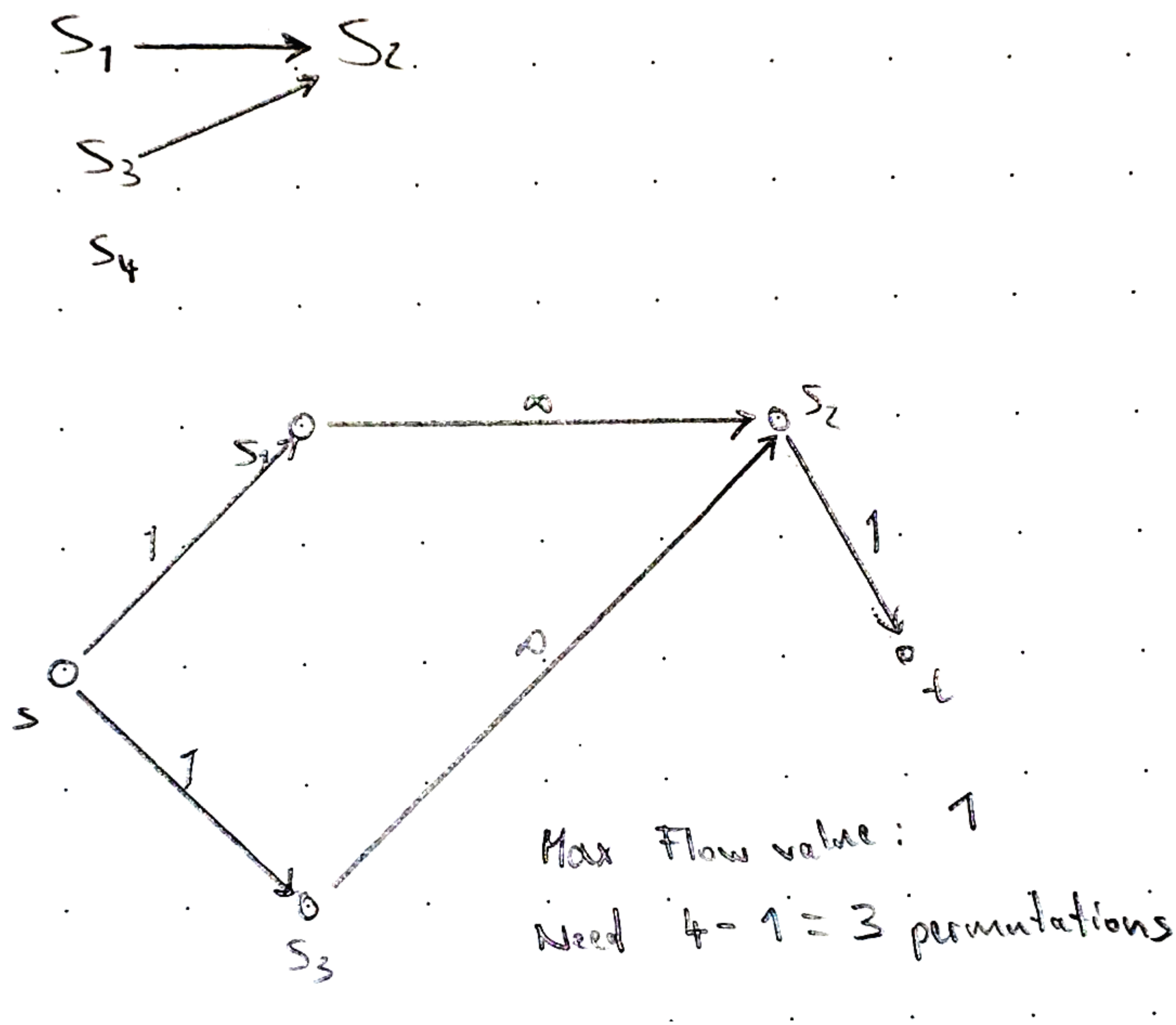
- If a vertex $v_{ij} \in A$, then $v_i \in A$ and $v_j \in A$. Otherwise, the minimum cut value would be infinite, because the capacities of the connecting edges are infinite.
- Therefore, if $v_i \notin A$ or $v_j \notin A$, then $v_{ij} \in B$.
- If $v_i \in A$ and $v_j \in A$, then $v_{ij} \in A$. Otherwise, we could decrease the minimum cut value by adding v_{ij} to A (decreasing the cut value by 1 for the edge (s, v_{ij})).
- We therefore know that the only cut edges are edges (v_i, t) and (s, v_{ij}).
- We can conclude that the edges (v_i, t) are relevant for this problem, because they define the bottleneck that prevents the algorithm from pushing more flow to the sink using a *team* vertex. I.e., we could increase the maximum flow by pushing more flow using these vertices but then we would eliminate a team. The edges (s, v_{ij}) are not relevant here, because we cannot increase them without adding additional edges in the graph G (multi-edges).

Runtime

- For the unoptimized version, we run the algorithm of subproblem a $\mathcal{O}(\frac{|E|}{\frac{1}{n^2}}) = \mathcal{O}(|E|n^2) = \mathcal{O}(|E||V|^2)$ times. This is the number of times we can increase α.
- The overall runtime complexity for the unoptimized version is therefore $\mathcal{O}(|E||V|^2 \cdot (|V| + |E|)^3)$.
- The overall runtime complexity for the optimized version with binary search is $\mathcal{O}(\log(|E||V|^2) \cdot (|V| + |E|)^3)$.
- There is most likely a much lower bound on the complexity, but the problem description just asked for a polynomial algorithm.

Problem 4

Basic Idea



- We begin with a preprocessing step: for every pair of sets S_i and S_j, we determine whether $S_i \subseteq S_j$ or $S_j \subseteq S_i$. Afterwards, we can form chains of subsets, e.g. $S_i \subseteq S_j \subseteq S_k$. In the figure above, we have the chains (S_1, S_2), (S_3, S_2) and (S_4). We represent these chains as a graph $G = (V, E)$ as shown in the upper part of the figure. There is an edge (S_i, S_j) if $S_i \subseteq S_j$.
- We generate a flow network for this graph. The flow network is a bipartite graph $G' = ((V_1' \cup V_2'), E')$ with $V_1' = \{v \in V \mid d_{in}(v) > 0\}$, $V_2' = \{v \in V \mid d_{out}(v) > 0\}$, where $d_{in}(v)$ is the incoming edge degree of v and d_{out} is the outgoing edge degree of v. We connect the left part with the soruce, and the right part with the sink. There is an edge (a, b) in the middle iff $a \subseteq b$.
- $k - maxFlow$ is the minimum number of permutations that we need.
- We can reconstruct the permutations, i.e. the subsets that correspond to one permutation by examining the flow values. Then, for every chain $(S_i, S_j, \ldots S_k)$, we can build the permutation in the

following order: all columns in S_i, all additional columns in S_j, ..., all additional columns in S_k, all other columns in the table.

Flow Graph Construction

- Add a source s and a sink t.
- For every $v_i \in V$, add a vertex $v_{in,i}$ if $d_{in}(v_i) > 0$.
- For every $v_i \in V$, add a vertex $v_{out,i}$ if $d_{out}(v_i) > 0$.
- For every $v_{in,a} \in V'$ and $v_{out,b} \in V'$, add an edge $(v_{in,a}, v_{out,b})$ with capacity ∞, if $a \subseteq b$.
- Add edges $(s, v_{in,a})$ with capacity 1.
- Add edges $(v_{out,b}, t)$ with capacity 1.

Algorithm

- For every pair S_i, S_j determine whether $S_i \subseteq S_j$ or $S_j \subseteq S_i$. This can be done by iterating over all elements in one set and checking if the element is present in the other set.
- Build the graph G: when adding the next vertex S_i, add an edge (S_k, S_i) if $S_k \subseteq S_i$ and add an edge (S_i, S_j) if $S_i \subseteq S_j$.
- Build the flow graph G' as described in the previous subsection.
- If $k - maxFlow > l$, output *no solution*.
- Otherwise, generate up to l lists L_i, where S_a and S_b are in the same list if $f((v_{in,a}, v_{b,out})) = 1$. Afterwards, every L_i contains a chain, and the union of all chains is Vi[10]. If a set S_c is not added to any list (this can happen, if there is no subset relation involving S_c), add it to a new list.
- Sort all chains by the $\subseteq$ relation, i.e. $S_a \in L_i \leq S_b \in L_i$ if $S_a \subseteq S_b$.
- Output a permutation for every L_i. The columns should be output in this sequence. Start with the first set in L_i and output all its columns. For the next set in L_i, output all columns that were not yet output for this L_i so far. Continue, until all sets are output. Output all columns of the table that were not output so far. Continue with the next L_i.

Proof

- We first show that the algorithm always generates a valid set of permutations, i.e. all permutations T_{p_i} together contain all sets S_i as prefixes.
- Every set S_i is contained in some list L_i, because there is either a subset relation $S_j \subseteq S_i$ or a subset relation $S_j \subseteq S_i$ (then we add both to the same list), or S_i is not part of a subset relation and we create a new list and S_i is added to that list.

[10] A set S_a can appear in more than just one L_i.

- After sorting the lists L_i, we can be sure that for $L_i = (s_1, s_2, \ldots, s_q)$ $s_{i-1} \subseteq s_i$. Therefore, if we generate the permutations in such a way that we first add the columns in s_1, then the columns in s_2, and so on, we can be sure that every s_i is a prefix of the permutation. We can prove this by contradiction. Let us assume that $s_i \subseteq s_j$ and s_j is a prefix of the permutation but s_i is not. In that case, there must be some column out of s_j in the permutation that is not in s_i before finishing the enumeration of all s_i. But this cannot happens, because we first add all columns of s_i and only then start adding additional columns from s_j.

- We can be sure that we actually generate a permutation for every L_i, because we add only new columns and fill up the permutation with missing columns from the table at the end.

- Through the minimum cut we can prove that the algorithm generates the minimum number of permutations. Therefore, we have to find the bottleneck in the network flow graph. We can be sure that, if $v_{in,a} \in A$, then $v_{out,b} \in A$ if $S_a \subseteq S_b$, because these vertices are connected with an infinite edge. On the other hand, if $v_{out,b_1} \in A$ and $v_{out,b_2} \in A$, then $v_{in,a} \in A$ if $S_a \subseteq S_{b_1}$ and $S_a \subseteq S_{b_2}$ because, otherwise, we could reduce the max flow by 1 by adding $v_{in,a}$ to A. We can show that the min cut value is the number of sets, for which we can reuse or extend an existing chain. Therefore, $k - \mathit{maxFlow}$ is the number of times we have to use a new chain. This obviously has to be true for at least one set, because we need at least one permutation.

Runtime

- Determining subset relation for every pair of sets: we have to check if each element is present in the other set. This can be done in $\mathcal{O}(n \cdot k^2)$.

- Building the graph G: we add k vertices and up to k^2 edges, therefore the runtime is $\mathcal{O}(k^2)$.

- Building the flow network G': we add $2k$ vertices and up to $(2k)^2$ edges, therefore the runtime is $\mathcal{O}(k^2)$.

- The max flow can be determined with the Edmonds-Karp algorithm in $\mathcal{O}(|V||E|^2) = \mathcal{O}(k^5)$.

- Creating the lists L_i: we add up to $\mathcal{O}(2k^2)$ vertices to the lists L_i.

- Sorting all L_i: there are no more than l lists, so the runtime is $\mathcal{O}(l \cdot (k^2 \log k^2)) = \mathcal{O}(k \cdot (k^2 \log k^2))$ because $l < k$.

- Output all permutations: there are $\mathcal{O}(k^2)$ sets in total in all L_i, and every set can contain up to n elements. Therefore, the runtime is $\mathcal{O}(nk^2)$. To check whether we already output a column, we can use a hash set with constant insertion/accessing runtime (for n elements: no more than $\mathcal{O}(n)$).

- The overall runtime of the algorithm is $\mathcal{O}(\max\{n \cdot k^2, k^5\})$.