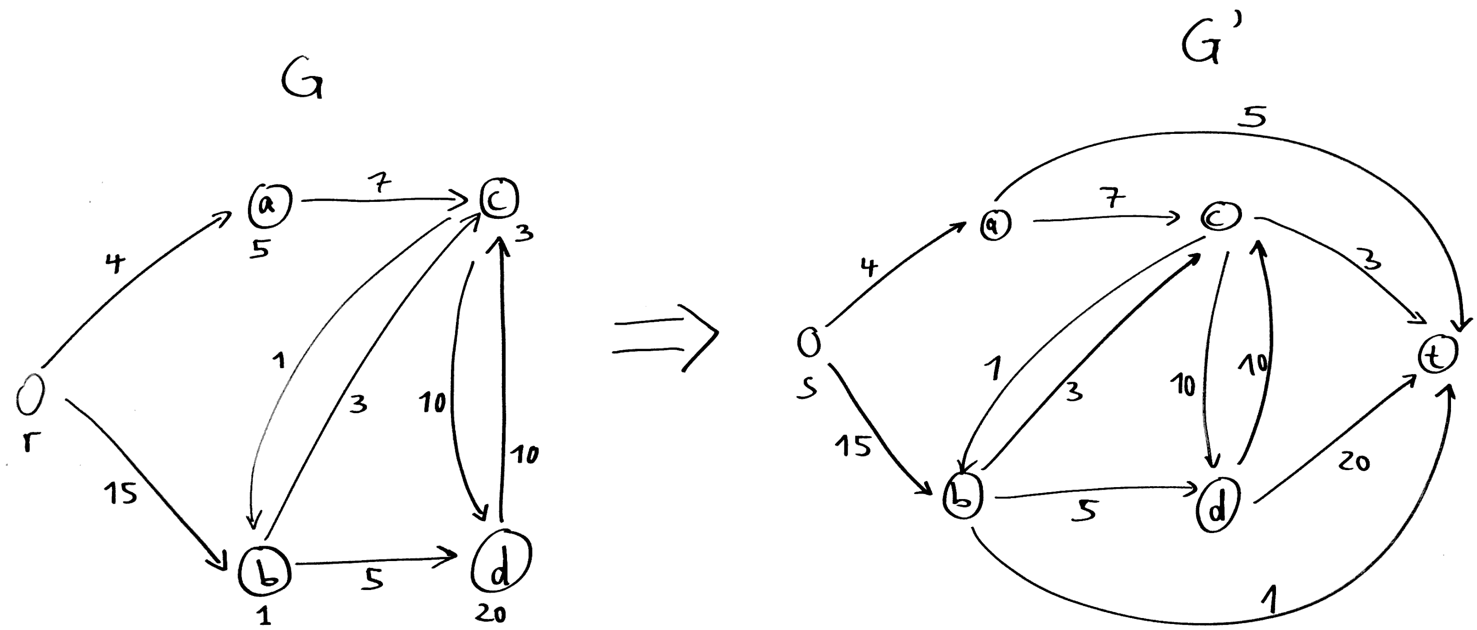
Problem 1: Maximizing the benefit of unreachable nodes

Basic Idea



- Build a network flow graph G' and calculate the minimum cut.
  - Take the original graph.
  - Connect every vertex $v \in V$ to the sink using a capacity of b_v. Adding such an edge to the cut means not to disconnect v and not to get the benefit v.
- Cut all edges that are in the minimal cut and part of the original graph.

Intuition

- We want to maximize the benefits of all disconnected vertices minus the cost for cutting the edges.
- Let A, B a partitioning of V, such that $r \in V$.
- maximize $\sum_{b \in B} v_b - cap(A, B) \Leftrightarrow$ minimize $cap(A, B) - \sum_{b \in B} v_b \Leftrightarrow$ minimize $cap(A, B) + \sum_{a \in A} v_a$
- Restate the problem: minimize the costs for cutting edges plus the benefits that we do not get.

Network Flow Construction

- Start with the original graph $G = (V, E)$.
- Let the special vertex r be the source $r = s$ and add an additional sink vertex t.
- For every vertex $v \in V - \{s, t\}$, add an edge (v, t) called e_v with capacity v_b.
- Call the new graph $G' = (V', E')$.

Algorithm 1 Partitioning/cutting V in A and B, such that $\sum_{b \in B} v_b - \textit{capacity}(A, B)$ is maximized in G

Require: Graph $G = (V, E)$
function MaxBenefit(G)
    $G' \leftarrow$ build flow graph out of G
    $A, B \leftarrow$ calculate minimum cut in G'
    return all cut edges e with $e \in E$
end function

Full Algorithm

In this algorithm, we can calculate the minimum cut by running the Ford-Fulkerson algorithm or the Edmonds-Karp algorithm and determining which vertices are reachable from the source. These vertices form the set A. The cut edges are all edges $e = (u, v) \in E$ with $u \in A$ and $v \notin A$.

Proof

We prove that the set of edges that are output by the algorithm, maximizes the profit if all these edges are removed.

- *Termination:* We assume that all b_v and c_e are rational numbers. Then, the Ford-Fulkerson algorithm is guaranteed to terminate.
- Reformulate the problem: We reformulate the original problem several times until we reach the min-cut formulation and show that the new formulation is equivalent to the previous one (if not trivial).
  - *Original problem:* Find a subset of edges $S \subseteq E$, such that $\sum_{b \in B} v_b - \sum_{e \in S} c_e$ is maximized, where B is the set of vertices that is no longer reachable from r.
  - Find $S \subseteq E$, such that $\sum_{e \in S} c_e - \sum_{b \in B} v_b$ is minimized.
  - Find $S \subseteq E$, such that $\sum_{e \in S} c_e + \sum_{a \in V - B} v_a$ is minimized. B and $V - B$ are complementary sets, i.e. maximizing $\sum_{b \in B} v_b$ is the same as minimizing $\sum_{a \in V - B} v_a$.
  - Find a partitioning of V in B and $V - B = A$, such that $\sum_{e=(u,v) \in E: u \in A \wedge v \in B} c_e + \sum_{a \in A} v_a$ is minimized. $V - B = A$ is the set of vertices that is reachable from r. The choice of A and B determines the choice of edges S (and vice-versa), i.e. the mapping from A/B to S is bijective. Given a set S or A/B, we can also reconstruct the other set A/B or S. Therefore, it does not matter whether we select S or A/B.
  - *Model vertices as edges:* Generate flow network graph G'. Find the minimum cut. In the min-cut of G', there must be no connection from s to t. For every $v \in V$ in G, we either have to cut $e_v = (v, t)$ with capacity b_v in G' or cut edges, such that $v \in B$. If $v \in A$, we have to pay the cost for cutting e_v. Therefore, this problem is equivalent to the previous one.

Runtime Complexity

- Building the flow network: we generate $\mathcal{O}(|V|)$ vertices and $\mathcal{O}(|E| + |V|)$ edges, where $|V|$ is the number of vertices in G and $|E|$ is the number of edges in G.
- Calculating the max-flow with the Edmonds-Karp algorithm: $\mathcal{O}(|V'||E'|^2) = \mathcal{O}(|V'|^5)$ (worst case: fully connected graph, i.e. $|E'| = \mathcal{O}(|V'|^2)$. In terms of G, calculating the max-flow takes $\mathcal{O}(|V|^5)$ time.
- Retrieving the min-cut/partitioning: run DFS from r in $\mathcal{O}(|V'| + |E'|) = \mathcal{O}(2|V| + |E|) = \mathcal{O}(|V| + |E|)$.
- Overall runtime complexity: $\mathcal{O}(|V|^5)$.

Problem 4: Approximating the independent set

LP relaxation

- Maximize $\sum_{v \in V} x_v$
- subject to
  - $\forall e = (u, v) \in E : x_u + x_v \leq 1$
  - $\forall v \in V : x_v \geq 0$

Polynomial time algorithm

In this section, we give a polynomial time algorithm that either finds an independent set of at least $\frac{|V|}{3}$ in G or certifies that no independent set of size greater or equal to $\frac{2|V|}{3}$ exists.

Basic Idea

- We describe a 2-approximation algorithm of the minimal vertex cover problem using linear programming and constraint the vertex cover to be at most of size $\frac{|V|}{3}$ in the relaxed problem.
- If the LP is feasible, we round the values. We will end up with a vertex cover of size $\leq \frac{2|V|}{3}$, resulting in an independent set of size $\geq \frac{|V|}{3}$.
- If the LP is infeasible, we can show that no independent set of size $\geq \frac{2|V|}{3}$ exists.

LP relaxation and 2-approximation for Vertex Cover

The minimal vertex cover problem can be written as a relaxed linear program as follows.

- *Minimize* $\sum_{v \in V} x_v$
- subject to
  - $\forall e = (u, v) \in E : x_u + x_v \geq 1$
  - $\forall v \in V : x_v \geq 0$

We can approximate the minimal vertex cover with a factor of 2 as follows.

- For every $v \in V$, set x_v to 1 if $x_v \geq 0.5$ (round up) and set x_v to 0 if $x_v < 0.5$ (round down).
- $T_{\text{LP}} = \{v \,|\, v \in V : x_v = 1\}$ is a vertex cover, because, for every edge $e = (u, v) \in E$, at least one of value of x_u and x_v will be 1. Because of the first constraint in the linear program, $x_v u$ and x_v can never be both smaller than 0.5 in the relaxed version. Therefore, at least one of them is rounded to 1 and, therefore, every edge is covered.
- T_{LP} is a 2-approximation. The objective function value of the relaxed LP is a lower bound for the size of the minimal vertex cover[1]. In the worst case, all $x_v = 0.5$ in the relaxed version. Therefore, we round all x_v to 1, resulting in an upper bound of $|V|$ for the approximation. Therefore, the approximation factor is $\frac{|V|}{0.5|V|} = 2$.

[1] If there was a smaller vertex cover, then the linear program would have found it (or an even smaller solution with fractions).

Duality of Vertex Cover and Independent Set[2]

Let $G = (V, E)$ be a graph. Then S is an independent set if and only if $V - S$ is a vertex cover. Let $e = (u, v) \in E$ be an arbitrary edge. u and v cannot be part of S at the same time, so $V - S$ is a vertex cover. Assume that $V - S$ is a vertex cover and let $e = (u, v) \in E$ be an arbitrary edge with $u \in S$ and $v \in S$. Then, neither $v \in V - S$, nor $u \in V - S$, contradicting our assumption that $V - S$ is a vertex cover. Therefore, u and v cannot be part of S at the same time and S is an independent set.

Full Algorithm

Algorithm 2 Finding an independent set of size at least $\frac{|V|}{3}$ or stating that no independent set of size at least $\frac{2|V|}{3}$ exists.

Require: Graph $G = (V, E)$

function ONETHIRDINDEPENDENTSET(G)
    $P \leftarrow$ relaxed LP for the minimal vertex cover for G
    Add constraint $\sum_{v \in V} x_v \leq \frac{|V|}{3}$ to P
    Solve P with the Ellipsoid method
    if P is infeasible **then**
        return No independent set of size $> \frac{2|V|}{3}$
    else
        return $\{v \mid v \in V : x_v < 0.5 \text{ in } P\}$
    end if
end function

Proof

- Let P be the relaxed LP of the minimum vertex cover problem with the constraint that the vertex cover must be at most of size $\frac{|V|}{3}$. Let x_v be the decision variable values in its solution.
- Case 1: P is feasible.
  - Let $\tilde{x}_v$ be the rounded values of x_v. Let $C = \{v \in V \mid \tilde{x}_v = 1\}$.
  - C is a 2-approximation of the minimal vertex cover in G[3].
  - $\sum_{v \in V} x_v \leq \frac{|V|}{3} \Rightarrow \sum_{v \in V} \tilde{x}_v = |C| \leq \frac{2|V|}{3}$
  - Let $S = V - C$. S is an independent set and $|S| \geq |V| - \frac{2|V|}{3} = \frac{|V|}{3}$.
- Case 2: P is infeasible.
  - G has no vertex cover of size $\leq \frac{|V|}{3}$. Otherwise, the LP would have found an assignment of x_i with such an objective function value.
  - Let C be the minimum vertex cover in G and S be the maximum independent set in G.
  - $|C| > \frac{|V|}{3} \Rightarrow |S| \leq |V| - |C| = \frac{2|V|}{3}$.
  - There is no independent set of size $> \frac{2|V|}{3}$. ■

[2] Proof taken from Kleinberg, Tardos textbook, page 455.
[3] Proof: see approximation section.

Runtime Complexity

- Building the linear program: we create $|V|$ variables, add $|V|$ non-negativity constraints and $|E|$ edge constraints. Therefore, the size of the linear program is polynomial in the size of G.
- The Ellipsoid method solves the linear program in polynomial time.
- The overall runtime complexity of the algorithm is polynomial in the size of G.

Polynomial time algorithm for graphs of degree 3

Basic Idea

- The idea from the previous subproblem can be generalized: given an ϕ-approximation algorithm of the vertex cover, we can came up with an algorithm that
  - finds an independent set of size at least $\alpha|V| = \frac{|V|}{1+\phi}$ or
  - or certifies that no independent set has size greater than $(1-\alpha)|V| = \frac{\phi|V|}{1+\phi}$
- The quality of the vertex cover approximation determines the value α.

Full Algorithm

The following algorithm is a general algorithm that works with any ϕ-approximation of the minimum vertex cover problem and yields an $\alpha = \frac{1}{1+\phi}$.

Algorithm 3 Finding an independent set of size at least $\alpha|V|$ or stating that no independent set of size at least $(1-\alpha)|V|$ exists.

Require: Graph $G = (V, E)$

function αINDEPENDENTSET(G)
    $\beta \leftarrow 1 - \alpha$
    $S \leftarrow \phi$-approximation for minimum vertex cover(G)
    if $|S| > \beta$ **then**
        return No independent set of size $> (1-\alpha)|V|$
    else
        return $V - S$
    end if
end function

Proof

- The run of the ϕ-approximation algorithm for the minimum vertex cover on G can have two possible outcomes. Let S be the ϕ-approximated minimum vertex cover.
- Case 1: $|S| \leq \beta|V|$
  - The approximated minimum vertex cover has size $|S| \leq \beta|V|$.
  - There is an independent set of size $\geq (1-\beta)|V|$.
- Case 2: $|S| > \beta|V|$

  - The approximated minimum vertex cover has size $> \beta|V|$. Therefore, the real minimum vertex cover has size $> \frac{1}{\phi}\beta|V|$
  - The maximum independent set has size $< (1 - \frac{\beta}{\phi})|V|$. Therefore, there is no independent set of size $\geq (1 - \frac{\beta}{\phi})|V|$.

- We must set $\beta = (1 - \frac{\beta}{\phi}$[4], otherwise, we end up with an interval between these two terms, where we cannot say whether the independent set exists or not. For a given ϕ, $\beta = (1 + \phi^{-1})^{-1} = \frac{\phi}{\phi+1}$.
- For $\alpha = 1 - \beta$, we get relations in terms of the independent set that match the previously mentioned formulas in the *Basic Idea* section.
  - Find an independent set of size at least $\alpha|V| = (1 - \beta)|V| = (1 - \frac{\phi}{\phi+1})|V| = \frac{|V|}{\phi+1}$ or
  - certifies that no independent set of has size greater than $(1 - \alpha)|V| = \beta|V| = \frac{\phi}{\phi+1}|V|$
- The quality of the the approximation determines the value of α that we get. For example, for the 2-approximation in the previous subproblem, we get $\alpha = \frac{1}{\phi+1} = \frac{1}{3}$. By finding a better approximation of the minimum vertex cover, we can increase α. ■

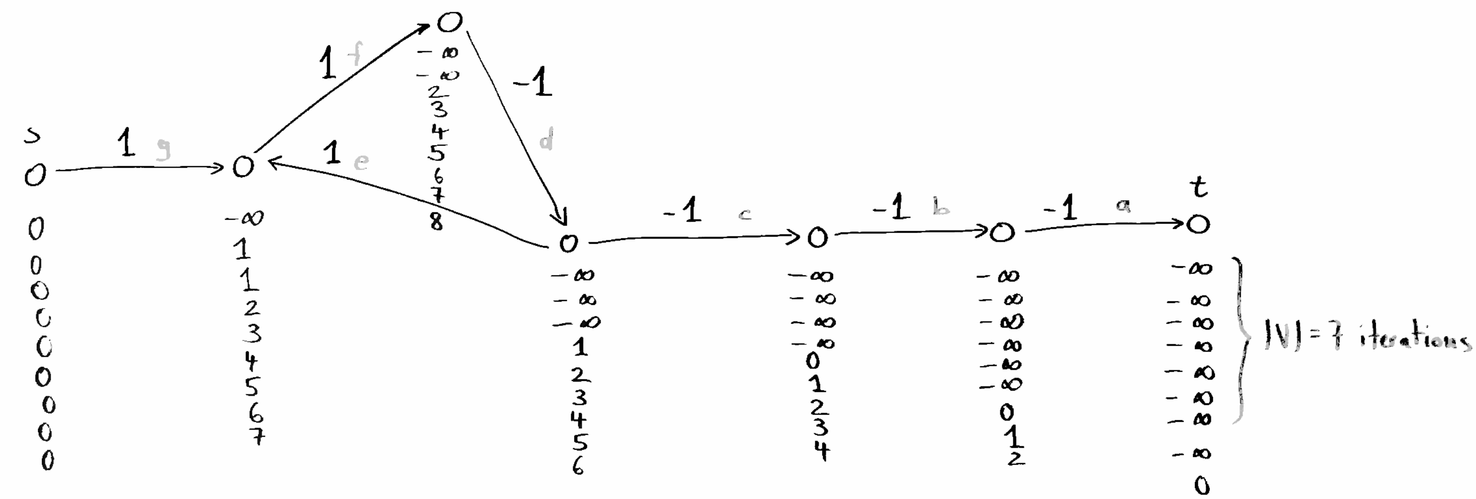
$\frac{3}{2}$-approximation for Vertex Cover

- Given an ϕ-approximation algorithm, we can immediately calculate a value of α and give an algorithm that finds an independent set of size $\alpha|V|$ or certifies that no independent set has size greater than $(1 - \alpha)|V|$, according to the previous argument.
- Algorithm: select the vertex with the highest degree, add it to the vertex cover, and remove it.
- For graphs with a maximum degree of 3, this is a $\frac{3}{2}$-approximation.
- In the worst case, the graph consists of cliques of size 4, i.e. every vertex in every clique has 3 connections. In that case, the minimum vertex cover is $\frac{3|V|}{4}$, because only 3 vertices cover all 6 edges per clique.
- The minimum number of vertices that a vertex cover must cover is $\frac{|V|}{4}$. Consider the case, where the graph consists of disconnected components of 4 vertices, where one vertex is in the middle and the other 3 vertices are connected to only this middle vertex. INSERT MORE PROOF HERE.
- Therefore, according to the previous argumentation and proof, we get an algorithm with $\alpha = \frac{1}{1+\frac{3}{2}} = \frac{2}{5}$.

[4]This is equivalent to $1 - \beta = \frac{\beta}{\phi}$.

Problem 5: Always non-negative path

Basic Idea



- The algorithm is similar to the Bellman-Ford algorithm. Iterate over all edges $e = (u, v) \in E$, $|V|$ times, and update $maxSum[v]$, i.e. the maximum achievable sum on an always non-negative s-v path.
- $\forall e = (u, v) \in E$: $maxSum[v] \leftarrow \max\{maxSum[v], maxSum[u] + w(e) \textbf{ if } maxSum[u] + w(e) \geq 0\}$, where $w(e)$ is the weight of edge e. Repeat this step $|V|$ times (*Bellman-Ford* step).
- As we can see in the illustration above, we might have to cycle in positive-weight cycles multiple times, in order to accumulate a value that is big enough to make up for a long sequence of -1 edges on the rest of the s-t path.
- We find the first reachable strictly-positive-weighted cycle, i.e. a positive-weight cycle C with $\forall v_C \in C : maxSum[v_C] \geq 0$, on an s-t path using DFS (if there is such a cycle) and set all $maxSum[v_C] = \infty$[5].
- We repeat the Bellman-Ford step, updating the *maxSum* values and using strictly-positive-weighted loops arbitrarily often if necessary.
- There is an always non-negative s-t path if and only if $maxSum[t] \geq 0$.

Full Algorithm

- The Bellman-Ford step is very similar to the the Bellman-Ford algorithm. Instead of trying to reduce the path length, we try to find a path weight that is as big as possible. We only use vertices with a non-negative path length, ensuring that this condition holds for subpaths starting from s.
- The DFS step runs a depth-first search using a stack. We somehow need to store, whether a vertex was already visited, e.g. by using an array. Once we reach an already visited vertex v, we compare the sum that we accumulated so far with $maxSum[v]$. If the accumulated sum is greater, then we found a strictly-positive-weighted cycle and increase all vertices on that cycle[6] to infinity. We do not visit vertices u with a negative value of $maxSum[u]$, ensuring that we only visit vertices that can be reached by an always non-negative path.

[5] It is crucial that all *maxSum* values are ≥ 0, because we would not be allowed to use the cycle in the first place if it cannot be reached by any always non-negative path.

[6] The cycle vertices are all vertices on the stack up to the vertex v, i.e. all vertices on the stack between v and the current vertex (including these vertices).

Algorithm 4 Deciding whether there is an always non-negative s-t path in G.

Require: Graph $G = (V, E)$

```
function AlwaysNonNegative(G)
    ∀v ∈ V : maxSum[v] ← −∞
    maxSum[s] ← 0
    BellmanFordStep()
    DfsStep()
    BellmanFordStep()
    return maxSum[t] ≥ 0
end function
```

Algorithm 5 Adapting *maxSum* for all $v \in V$ by propagating all these values using every edge $|V|$ times.

```
function BellmanFordStep()
    for i ← 1 to |V| do
        for all e = (u, v) ∈ E do
            if maxSum[u] + w(e) ≥ 0 then
                maxSum[v] ← max{maxSum[v], maxSum[u] + w(e)}
            end if
        end for
    end for
end function
```

Algorithm 6 Setting $maxSum[v] \leftarrow \infty$ for all $v \in C$ for at least the first reachable strictly-positive-weighted cycle C on every always non-negative s-t path.

```
function DfsStep()
    S ← new Stack
    maxSum_old ← copy(maxSum)
    S.push((s, 0))
    while |S| > 0 do
        (v, m) ← S.peek()
        mark v as visited
        for all u ∈ V : (v, u) ∈ E do
            if maxSum_old ≥ 0 then
                if u was already visited then
                    if u ∈ S ∧ m + w(e_{v,u}) > maxSum_old[u] then
                        for all a ∈ V : (a, x) ∈ S ∧ (a, x) is not before (u, y) in S do
                            maxSum[a] ← ∞
                        end for
                    else
                        S.push(u, m + w(e_{v,u}))
                    end if
                end if
            end if
        end for
        S.pop()
    end while
end function
```

Proof

Lemma 1 *After the first* BellmanFordStep*, for all* $v \in V$*,* $maxSum[v]$ *is the maximum sum of all always non-negative* s*-*v *paths of length* $\leq |V|$*, if such a path exists, and* $-\infty$ *otherwise.*

We prove this by induction over the number of used edges k, that after k iterations of the outer loop, $maxSum[v]$ is the maximum sum of all always non-negative s-v paths using at most k edges.

- *Induction Base Case:* For $k = 0$, we are not allowed to use any edge. Therefore, $maxSum[s] = 0$ and for all other vertices $v \in V - \{s\}$, $maxSum[v] = -\infty$.
- *Induction Hypothesis:* Let the statement be true for an arbitrary but fixed k.
- *Induction Step:* When we update $maxSum[v]$, for some $v \in V$ and $e = (u, v) \in E$, we set $maxSum = maxSum + w(e)$. We only do this if the new value of $maxSum[v]$ is non-negative. $maxSum[u] \geq 0$, because we never set $maxSum$ to a negative value (except for the initialization), and, by induction, $maxSum[u]$ is the maximum sum of all always non-negative s-u paths using $k - 1$ edges. In every iteration, we try to improve paths using all possible edges, and only update $maxSum$ if it becomes greater. Therefore, at the end of kth iteration, $maxSum[v]$ is the maximum sum of all always non-negative s-v paths using at most k edges. If $maxSum[v] = -\infty$, it was not updated because v is not reachable from s using an always non-negative path with at most k edges.

Lemma 2 *Let* P *be an arbitrary* s*-*t *path, and* C *be a strictly-positive-weighted cycle* C *with* $P \cap C \neq \emptyset$*, such that* C *is the first such cycle for* P *and for all* $v_c \in C$*, there exists an always non-negative* s*-*v_c *path in* G*. After* DfsStep*, for all* $v_c \in C$*,* $maxSum[v_c] = \infty$*.*

When the DFS visits a vertex, it accumulates the edge sum for the current path starting from s. When it reaches an already visited vertex v and v is not on the stack anymore, then we found an undirected, but no directed cycle. If v is still on the stack, then all vertices starting from v to the top of the stack form a cycle C[7]. The algorithm takes all these vertices and sets their $maxSum$ values to ∞.

Let us assume that a vertex $v_c \in C$ is not reachable by an always non-negative s-v_c path in G. Then, at least one vertex u on any s-v_c path has $maxSum[u] = -\infty$. Then, the DFS will not visit this vertex and there is no way to find the cycle, because, either a vertex on the path to the cycle is not visited, or a vertex inside the cycle is not visited.

Note, that this works only, because $maxSum[v]$ contains the maximum sum of all always non-negative s-v paths using at most $|V|$ edges. Since C is the first cycle on an s-t path, there is no way to accumulate a high edge sum in order to reach C with an always non-negative path. Therefore, if a cycle $|C|$ cannot be reached by an always non-negative path using at most $|V|$ edges (which can involve all vertices), then C cannot be reached by an always non-negative path at all.

[7] If there was a vertex $u \in V$ among these vertices, that is not part of the cycle, then the DFS would not have been able to continue the path to v and popped u from the stack.

Lemma 3 *There is an always non-negative s-t path in G, if and only if*

- *t can be reached from s via an always non-negative path of length at most $|V|$ or*
- *there is an always non-negative path of length at most $|V|$ to a strictly-positive-weighted cycle C, and some $v_C \in C$ has an arbitrary path to t.*

In the first case, the algorithm finds that path in the first BellmanFordStep according to the first lemma. In the second case, the algorithm find the first strictly-positive-weighted cycle C that insects with P, where P is some always non-negative s-t path. The distance C from s cannot be greater than $|V|$, because, otherwise, we would have to visit a vertex twice, which results in a cycle, contradicting our assumption that C is the first such cycle. After we reached the cycle C, we can loop in it as often as we want to and accumulate an arbitrary high *maxSum* value, therefore $maxSum[v_C] = \infty$. Therefore, t is reachable using an always non-negative path from v_C, regardless of the weights on that path. Using the same argument from the first lemma, we can prove that by running BellmanFordStep again, we update all $maxSum[u] = \infty$ for every vertex u on any v_C-t path. Therefore, the algorithm sets $maxPath[t] = \infty$ and outputs the correct answer.

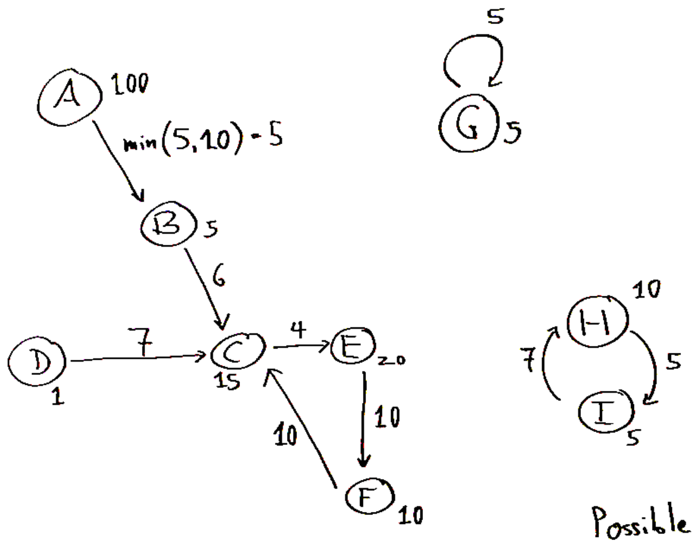
If none of the two cases applies, then there is no simple always non-negative s-t path and there is no way to increase the sum in loop that leads to t. Therefore, $maxSum[t] = -\infty$ after the first BellmanFordStep (according to the first lemma), and either DfsStep does not change *maxSum* values or there is no way from the loop to t, in which case the second BellmanFordStep cannot propagate the increased values to t. ∎

Runtime Complexity

- Running BellmanFordStep: $|V|$ iterations for the outer loop and $|E|$ iterations for every inner loop run, resulting in $\mathcal{O}(|V||E|)$ iterations.
- Running DfsStep: The runtime of DFS is $\mathcal{O}(|V| + |E|)$. In the worst case, we update no more than $|V|$ *maxSum* values per vertex, resulting in $\mathcal{O}(|V|^2 + (|V| + |E|)$.
- Running BellmanFordStep again: $\mathcal{O}(|V||E|)$.
- The overall runtime complexity is $\mathcal{O}(|V||E| + |V|^2 + |V| + |E| + |V||E|)$. In the worst case, $|E| = |V|^2$ (fully connected graph), so the runtime complexity is $\mathcal{O}(|V|^3)$.

Problem 2: Coupon Collector

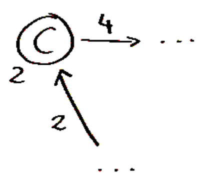
Basic Idea



Variety	Price	Coupon	Discount
A	100	B	10
B	5	C	6
C	15	E	4
D	1	C	7
E	20	F	10
F	10	C	10
G	5	G	5
H	10	I	5
I	5	H	7

Possible output of algorithm: A, B, D, G, I, H, C, E, F

After buying A, B, D:



- Model the problem as a graph. Every variety i gets a vertex $w_i = v_i$ with weight p_i. Add an edge (v_i, v_j) with weight $w_{i,j} = \min\{p_j, v_i\}$, indicating that we might get a discount of $w_{i,j}$ if buy variety i before variety j.
- *Intuition:* Always buy the variety such that we will loose the smallest discount. After buying variety i with a coupon for j, subtract $w_{i,j}$ from w_j, indicating that j is now cheaper to buy. Update j's incoming edges, such that their weight is not bigger than w_j (you cannot save more money than the variety costs)[8].
- Buy all varieties with no incoming edges first (we will loose no discount and we have to buy them anyway), until there are only cycles left (see proof).
- For every cycle C, buy the variety i with $v_i \in C$, where $w_{j,i}$ is minimal and $(v_j, v_i) \in E$ (at this time, there can only be one incoming edge for every vertex), i.e. buy the variety where we loose the least discount. Buy the rest of the varieties in C by following the edges in the cycle.

Graph Construction

- For every variety i, add a vertex v_i with a weight $w_i = v_i$.
- If variety i contains a coupon for variety j with discount v_i and the regular price of j is p_j, add an edge (v_i, v_j) with weight $w_{i,j} = \min\{p_j, v_i\}$.

[8] It is sufficient to update all edges at once only a single time after buying all varieties with no incoming edges, leading to linear runtime.

Full Algorithm

Algorithm 7 Coupon Collector Algorithm

```
function CouponCollector()
    build graph G
    for all v ∈ V do
        if v was not yet visited then
            DfsBuy(v)
        end if
    end for
    ∀e = (u, v) ∈ E : w_{u,v} ← min{w_{u,v}, w_v}
    for all v ∈ V do
        s ← vertex u in v's cycle such that w_{x,u} is minimal for some x
        while s.next ≠ null do
            buy(s)
            s ← s.next
            delete(s)
        end while
    end for
end function
```

Algorithm 8 Buy a variety and update the prices and achievable discounts

```
Require: Vertex v
function DfsBuy(v)
    mark v as visited
    if deg_in(v) = 0 then
        buy(v)
        delete(v)
    end if
    if v.next was not yet visited then
        DfsBuy(v.next)
    end if
end function
```

- Every vertex has exactly one outgoing edge, since every variety contains exactly one coupon. The next pointer points to the next variety.
- DfsBuy buys all varieties with no incoming edge, i.e. all varieties which we will never get a coupon for. After buying a variety, we delete its vertex from the graph G and thus also from the vertex set V.
- CouponCollector calls DfsBuy. Afterwards, there will be only disjunct cycles left (see proof). For every cycle C, we start buying the whole cycle by following the next pointers, starting with the vertex whose incoming edge has the least value, i.e. the variety for which we will loose the smallest discount.
- Buy buys a variety, removes it from the graph, and updates price values and discount values. After buying a variety, the price for the variety of the coupon j drops.

Algorithm 9 Buy varieties with no other incoming edges

Require: Vertex v

```
function BUY(v)
    w_{v.next} ← w_{v.next} − w_{v,v.next}
    buy variety(v)
    delete(v)
end function
```

Runtime Complexity

- Generating the graph G: we generate exactly n vertices and n edges. This takes $\mathcal{O}(n)$ time.
- Running DfsBuy: we call this function for every vertex v_i exactly once. Runtime $\mathcal{O}(n)$, since the number of edges is also n.
- Updating all edge weights: there are no more than n edges, so the runtime complexity is $\mathcal{O}(n)$.
- Buying the rest of the varieties: with every run of the for-loop, V becomes smaller. We run the for-loop number of cycles times. Every run of the for-loop eliminates a cycle. Inside a cycle C, we find the vertex for which we loose the minimum discount by following the next pointer no more than $|C|$ times (full loop). Then we traverse the C a second time and buy every vertex. The runtime for one cycle is $\mathcal{O}(|C|)$. Since the cycles are disjunct, the runtime for the whole step is $\mathcal{O}(n)$.
- Every variety is bought once. The runtime complexity of all Buy steps is $\mathcal{O}(n)$
- The overall runtime complexity of the algorithm is $\mathcal{O}(n)$.

Proof

Lemma 4 *Varieties with no incoming edges can be bought in any order (*DfsStep*) and it is an optimal decision to buy them first. Repeating this step until no such vertices exist, is an optimal decision.*

We can never get a discount for varieties with no incoming edges, since there are no coupons for these varieties. We still have to buy these varieties. Therefore, we can buy these varieties in any order (they are independent of each other). Furthermore, it is safe to buy this elements first. Consider an optimal buying sequence S_{OPT} and let i be a variety with no incoming edges. By moving i to any other position, the cost of the sequence does not change, because we will never get a discount for i.

Let j be a variety that ends up with no incoming cycle edge after the previous buying step. The same argument holds true, as long as we make sure that we buy j after i, where there used to be an edge (i, j) before buying and deleting i (the cost of S_{OPT} does not change by buying j at a different time, as long as we buy it before i). Therefore, by induction, we can prove that recursively buying all varieties with no incoming edges, until no such variety exists anymore, is an optimal decision.

Lemma 5 *If there are no vertices with no incoming edges, G consists of disjunct cycles.*

Let $v \in V$ be an arbitrary vertex. v has exactly one outgoing edge, denoted by v.next. By following the next pointer we will eventually reach a vertex that we already visited (loop L_1), since the number of vertices is finite. The first already visited vertex that we reach must be the vertex v that we started with.

Let us assume that we reach another vertex $u \neq v$ that we already visited. Then, by traversing G starting from v using inverted edges, we cannot reach any other already visited vertex (otherwise, we would have reached v instead of u first). We cannot end up at a vertex with no more incoming edges, by definition of G. Therefore, since the number of vertices is finite, we will run into another loop L_2, where no vertex is an already visited vertex. Since, every vertex in L_2 contains a pointer to the next element in L_2, and since at the same time, we can reach v and thus L_1 from L_2, there must be a vertex in L_2 with two outgoing edges. This contradicts the fact that every vertex has exactly one outgoing edge.

Lemma 6 *Let G be a graph and assume that we already bought all varieties with no incoming edges. Let, for every variety i, v_i be the price for buying i with coupons that we go so far, and for every $(v_i, v_j) \in E$, $w_{i,j} \leq w_j$. Buying the varieties in every cycle in such a way that we loose the least discount is an optimal decision. Every such sequence is optimal, no matter how the sequences for every cycle are interleaved.*

Let C be a cycle. We have to break the cycle at one position, i.e. when we buy the first variety $v \in C$, we will not get the discount for v. We will get the discount for buying all subsequent varieties in the loop by following the next pointer. Since we cannot get the discount for at least one variety, $\max_{v \in C} \sum_{u \in C - \{v\}} w_{x,u}$ is an upper bound the total discount that we get, i.e. the optimal decision for C. We get exactly this total discount for C if we start with buying a variety v with a minimal value of $w_{x,v}$ in G. We get all other discounts by following the next pointer and buying the rest of the varieties in that sequence. Since, for every $(v_i, v_j) \in E$, $w_{i,j} \leq w_j$, we cannot get money back by using a coupon. Since the values of w_j are the prices that take into account discounts that we got during buying varieties with no incoming edges, this is an optimal decision for C, having already bought all varieties with no incoming edges (and this was shown to be optimal).

Let D be another cycle such that C and D are disjunct. For D we can make the same argument as for C. Now consider an optimal buying sequence S_{OPT}. In S_{OPT}, we buy varieties for cycle C and D as described above, i.e. for C and D, we start the variety for which we loose a minimal discount and then follow the next pointer (we have shown that this is optimal). C and D are disjunct and, since every vertex has exactly one outgoing edge, there are no connections between C and D. Therefore, the buying sequence for C is independent from the buying sequence for D. Therefore, as long as C and D are bought in an optimal sequence, it is irrelevant how C and D are interleaved. Therefore, for an arbitrary number of cycles, as long as every cycle is bought in an optimal way, every interleaving of the buying sequences is optimal.

We have shown that buying varieties with no incoming edges is an optimal decision. We have also shown that buying the rest of the cycles as described in the algorithm, is an optimal decision, if we update the discount values after the first step in such a way that they reflect the coupons that we got when bought varieties with no incoming edges. Therefore, the algorithm produces an optimal solution. ■

Problem 6: Frequent Elements

Basic Idea

- Maintain two counter variables c_1, c_2, for the two most frequent elements at the moment.
- Read element by element: when element x is read, increase x's counter. If x there is no counter for x, decrease both counters by $\min\{c_1, c_2\}$ and make the variable with value 0 the new counter variable for x (and then increase x's counter).
- When all variables were read, iterate once over the whole array once again to make sure that the last most frequent elements actually apprear more than $\frac{n}{3}$ times.

Full Algorithm

See pseudo code listing on next page.

Runtime Complexity

We calculate the runtime complexity in terms of comparisons of elements.

- In the first for-loop, we compare every element no more than twice (with e_1 and e_2), resulting in $2n$ comparisons.
- In the second for-loop, we do the same kind of comparison again, resulting in $2n$ comparisons.
- In total, we have $4n = \mathcal{O}(n)$ comparisons.

Proof

We prove that, after reading all elements once, only the elements e_1 and e_2 may appear more than $\frac{n}{3}$ times.

- Let L be the list of elements.
- c_1/c_2 is the number of occurrences of e_1/e_2, since the time when e_1/e_2 replaced another number.
- Assume that there is another element $e \neq e_1$ and $e \neq e_2$ and e appears more than $\frac{n}{3}$ times. Let us think about why $e \neq e_1$ and $e \neq e_2$.
  - $e \notin L$, therefore e did never replace e_1 or e_2. Then, e does not appear more than $\frac{n}{3}$ times, which is a contradiction to our assumption.
  - $e \in L$, but it did never replace an element e_1 or e_2. e can only replace another element, if another element's counter drops to 0 and we read e. If this situation never happened, then some other elements that used to be in e_1 or e_2 did both appear more often than e. Therefore, e cannot appear more than $\frac{n}{3}$ times in L. Otherwise, there would be more than $\frac{2n}{3}$ other elements, resulting in more than n elements in total, but the list is only of size n.
  - $e \in L$ and $e = e_1$ or $e = e_2$ at some point, but it was replaced by another element g. We prove that, if n_e is the number of occurrences of e, there are at least $2n_e$ other elements. Assume, without loss of generality, that $e = e_1$.
    - If $c_1 = c_2$, then we need n_e elements to reduce e's counter to zero and we know that there are $c_2 = n_e$ occurrences of another element e_2. Therefore, there are at least $2n_e$ other elements.

Algorithm 10 Find all elements with that occur more than $\frac{n}{3}$ times

Require: List L

```
function FrequentElements(L)
    if |L| ≤ 2 then
        return L
    end if
    e_1 ← null
    e_2 ← null
    c_1 ← 0
    c_2 ← 0
    for all e ∈ L do
        if e_1 = e then
            c_1 ← c_1 + 1
        else if e_2 = e then
            c_2 ← c_2 + 1
        else if c_1 = 0 then
            e_1 ← e
            c_1 ← 1
        else if c_2 = 0 then
            e_2 ← e
            c_2 ← 1
        else
            c_1 ← c_1 − 1
            c_2 ← c_2 − 1
        end if
    end for
    c_1 ← 0
    c_2 ← 0
    for all e ∈ L do
        if e_1 = e then
            c_1 ← c_1 + 1
        else if e_2 = e then
            c_2 ← c_2 + 1
        end if
    end for
    return {e_i | c_i > n/3}
end function
```

  * If $c_2 < c_1$, then, at some point, we reduce e_2's counter to zero first. After this happened, we can read another symbol without having the counters reduced by one (since $c_2 = 0$). If we repeat this step with h different elements, we only reduce the counter n_e by $\frac{h}{2}$. Therefore, there are c_2 occurrences of e_2, we need c_2 occurrences of another symbol to reduce c_2 to 0, and then we have to read $2(n_e - c_2)$ symbols, in order to reduce e's counter ($n_e = c_1$) to zero[9]. Therefore, there are $2(n_e - c_2) + c_2 + c_2 = 2n_e$ occurrences of other symbols.
  * If $c_1 < c_2$, then there is a symbol e_2 that appears more often than e, i.e. more than n_e times, and there at least n_e other symbols that cause c_1 to drop to zero. Therefore, there are more than $2n_e$ other symbols in L.
  * In all three cases, there are $2n_e$ occurrences of other symbols. If e occurred more than $\frac{n}{3}$ times, then we have more than $\frac{2n}{3}$ other symbols. Therefore, we would have more than n elements in total, which is a contradiction to the fact that the list has size n. Therefore, by contradiction, e cannot appear more than $\frac{n}{3}$ times in L

- We proved that no other element than $e \neq e_1$, $e \neq e_2$ can appear more than $\frac{3}{n}$ times in L. Therefore, if an element e appears more than $\frac{n}{3}$ times, then either $e_1 = e$ or $e_2 = e$. Therefore, after counting the occurrences of e_1 and e_2 in a second run, we can be sure that we found all elements that appear more than $\frac{n}{3}$ times in L. ■

Space Complexity

- We need four variables c_1, c_2, e_1, e_2, resulting in constant space.
- If we also account for the input array n, we need $\mathcal{O}(n)$ space.

[9] This works by reading a symbol a and putting it in e_2, reading another symbol b and reducing both c_1 and c_2 at once, multiple times. If we read the same symbol again, i.e. $a = b$, then we again increase the counter of c_2 and we have to read one more other symbol to reduce any one of the two counters.

Problem 3: Scheduling

Example

For the following list of jobs, the heuristic generates a non-optimal solution.

- Jobs lengths: $\{15, 14, 13, 11, 10\}$.
- Assignment: $M_1 = \{15_1, 11_4, 10_5\}, M_2 = \{14_2, 13_3\}$[10], overall time requirement: 36.
- Optimal assignment: $M_1 = \{15, 14\}, M_2 = \{10, 11, 13\}$, overall time requirement: 34.

Basic Idea

- We find a lower bound for the optimal solution and an upper bound for the approximation algorithm, in order to determine the approximation factor.
- The key idea is to take a look at the machine with the highest load and at the last job that was added to that machine.

Proof

- Let G be greedy algorithm, M_1 be the first machine, M_2 be the second machine, $l(M_i)$ be the load of machine i, and t_i be the load of the ith job, where the jobs are sorted decreasingly by their load.
- Let, without loss of generality, M_1 be the machine with the higher load after running G.
- Let us assume that $n > 2$. Otherwise, we would give every machine at most one job which is optimal.
- $T_{OPT} \geq 2t_3$, one machine must get at least 2 jobs and in the best case, these two jobs have equal load. We also know that $T_{OPT} \geq \max_{1 \leq i \leq n} t_i$, because one of the two machines has to get the largest job. We also know that $T_{OPT} \geq \frac{1}{2} \sum_{i=1}^{n} t_i$, for the case that all jobs are equally distributed.
- Let us assume that M_1 gets at least 2 jobs. Otherwise, there is one big job that is greater than all other jobs (on M_2). In that case, the schedule of G is optimal.
- Let t_j be the last job that was assigned to M_1. We know that $j > 2$, because the first to jobs go to machines M_1 and M_2. Therefore, $t_j \leq t_3 \leq \frac{T_{OPT}}{2}$.
- $2(l(M_1) - t_j) \leq \sum_{i=1}^{n} t_i$, because M_1 had a lower load than M_2 when assigning t_j. Therefore, $l(M_1) - t_j \leq \frac{1}{2} \sum_{i=1}^{n} t_i = T_{OPT}$.
- Therefore, $l(M_1) \leq T_{OPT} + t_j \leq \frac{3}{2} T_{OPT}$, because $t_j \leq \frac{1}{2} T_{OPT}$ (and $l(M_1)$ biggest load of a processor after running G). This results in an approximation factor of $\frac{3}{2}$.

[10]The subscript indicates the time of assignment.